The Relevance Logic Newsletter

Volume 2 no. 2 May 1977

Editor: Robert G. Wolf

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The Office of Research and Projects, Graduate School, Southern Illinois University, Edwardsville, has given financial support to subsidize both the printing and mailing of this issue. We would like to express our gratitude for their generous support, especially to Vaughnie J. Lindsay, Dean, Graduate School, and to John A. Rodman, Director, Office of Research and Projects.

CALL FOR PAPERS

Studia Logica intends to devote its entire issue No. 2 or 3 of Vol. 37 (1978) to the topic: Relevance Logic. Articles and reviews touching on any aspect of the concept of relevance as it appears in logic are invited, with special emphasis on the tradition founded by Wilhelm Ackermann's concept of Strenge Implikation.

Papers should be sent to the guest editor for this number, Nuel D. Belnap, Jr., Department of Philosophy, University of Pittsburgh, Pittsburgh, PA 15260.

The deadline is Nov. 30, 1977. Because of definite size limitations, no paper should be more than 25-30 typewritten pages.

The following is quoted from a recent issue of Studia Logica: Manuscripts should be written in English. All manuscripts for publication should be typewritten with wide margin and wide spacing between the lines and should be sent in two copies. The Editors will arrange them for critical refereeing before acceptance. The authors should keep a complete copy of their manuscripts. Proofs will be sent to authors for correction. As a rule, the author will be submitted a proof of contribution for revision only once. The Editors assume the manuscript to be final version, any changes during galley-proofs are inadmissible. Authors receive 20 reprints of their papers free of cost.

ANNUAL CONFERENCE OF THE AUSTRALASIAN ASSOCIATION FOR LOGIC

CANBERRA, 1976

The 1976 Annual Conference of the Australasian Association for Logic was held at the Research School of Social Sciences, Australian National University, Canberra, Australia, from October 8-10. The Conference was co-sponsored by the *Research School for Social Sciences* and the *Association for Symbolic Logic*. Seventeen papers were given and abstracts of these papers are reprinted below in the order in which they were given. Richard Routley (President) and Michael McRobbie (Secretary) retired from the AAL executive and were replaced by Barry Taylor and Jan Gallé, both from the Department of Philosophy, University of Melbourne, who were elected as President and Secretary of the AAL respectively. The 1977 AAL Conference will be held at the University of Melbourne at a date to be announced.

M.A. McRobbie

PARADISE LOST

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The set F of finite subsets of the set N of natural numbers may be ordered $\emptyset,\{1\},\{2\},\{1,2\},\{3\},\ldots,\{1,2\ldots n\text{-}1\}$, $\{n\}\cup$ all preceding sets, ..., so F is denumerable. The set M of infinite subsets mentionable by finite expressions is known to be denumerable, so $F \cup M$ is denumerable.

Cantor's Theorem claims that in every 1 to 1 mapping from N to its power set $P(N)$, where $f(x)$ is the image of x, $\{x{:}x\in N \ \&\ x\notin f(x)\}$ is an unimageable element of $P(N)$ so that $P(N)$ is not denumerable. If so, each such element is unmentionable, but Cantor's "definition" purports to mention it, which is absurd. Hence the "definition" is non-significant.

The set C_A of all elements of N not belonging to their images in $P(N)$ is $\{x{:}x\in N \ \&\ x\notin f_A(x)\}$, where $N\cup\{A\}$ (for any item $A\notin N$) is 1 to 1 mapped on to $P(N)$, and $f_A(A) = C_A$. Alternatively, $C_y = \{x{:}x\in N-\{y\} \ \&\ x\notin f_y(x)\} = f_y(y)$ in a 1 to 1 mapping of N on to $P(N)$. In each case the element (A of $N\cup\{A\}$ or y of N) whose image (C_A or C_y) is defined is exempted from the defining condition. It can be shown that a 1 to 1 mapping can be made from $N-\{y\}$ on to $P(N)-\{N,C'\}$ for a suitably defined set C' whose condition does not exempt y. Each of these procedures shows that $P(N)$ is denumerable.

By neither invoking A nor exempting y nor taking the third way Cantor's "definition" both demands and rejects any proposed image y in N as an element. The contradiction signifies, not that there is some unimageable element of $P(N)$ (for the apparently intended set, either C_A or Cy or C', is otherwise specifiable and suitably imageable) but that the "definition", being incompatible with the mapping involved, is incoherent. To suppose otherwise leads to the paradox of mention.

THE STRUCTURE OF TYPE THEORY

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We call a theory a type theory when one of its central principles is that some or all of the world's individuals fall into types, and that it is somehow necessary that the characteristic features of individuals in any type are unique to that type. Type theories may be found in metaphysics, mathematics, logic, linguistics, biology, and elsewhere. The present paper is metaphysical rather than mathematical. Its aim is to discover some central formal principles of type theory, which ought to appear within the foundations of any reasonable type theory. We concentrate on the language-relative portion of this formal theory - that chunk of it which concerns the way a language L might reflect categorial features of the universe (if there are any). We propose principles governing ideas like these: (a) A type for L: a class of individuals all of which stand in exactly the same significance relations to the predicates of L. (b) Spanning in L: the significance relation between predicates of L and individuals. (c) A type predicate of L: a predicate which is (roughly) of necessity true in L of exactly a type. (d) The absolute value of a predicate P: a predicate true of just what P spans. (e) Type incompatibility between predicates. (f) A category for L: the extension of the absolute value of one of L's predicates. (g) A category predicate of L. We discover modality within typicality, consider the iteration of absolution, and intimate relevance within categoriality.

PHYSICAL TOPOLOGY: SOME PROBLEMS

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The paper is concerned with the ontological status of intervals of physical space. Some philosophers (e.g. Grünbaum) hold that the relation between points of physical space and intervals of physical space is that of membership, making intervals sets of physical points. There are at least two motives for adopting such a

position. One is a topological motive, so as to be able to use the set-theoretic machinery of topology to describe the topology of physical intervals. Another is a measure-theoretic motive. If one wants to talk sensibly about the size of an interval, and wants to use measure theory to do this, then (presumably, though this is disputed in this paper) one must identify intervals with sets, for in measure theory measures are defined only on sets. Furthermore, if one wants (as Grünbaum does) to avoid the question of the size of physical points, in either discrete or continuous space, then one can do this if one treats points as members of the things that have measures i.e. intervals.

We argue that set-theory is inadequte by itself to be able to say in what the topological structure of physical space consists. The argument is basically simple: a given set can in general have many topologies. If someone replies that physical space and intervals are not simply sets of spatial points, but topological spaces constructed from points, then that person must explain why one topology rather than another is selected, since the axioms of set theory guarantee that enough subsets of a given set exist to give us many topologies. The answer must lie in the fact that the points of space are ordered by relations which are not themselves sets, though these relationships between points are mirrored (almost, but not quite) exactly by a certain, set-theoretical topology. But this points the way to saying that intervals are wholes of points. Since we agree with this position, it is required of us to be able to say in what certain relationships between points consist, e.g. their continuity. Set theory is used to give several alternative accounts of the structure of the one-dimensional continuous physical line, without identifying intervals with sets or ordering relations with sets of ordered pairs. An analogue of Lebesgue measure is then specified, according to which intervals have non-zero size and points, which are parts of intervals, have zero size. Problems of multidimensional space are left unsolved.

TWO REMARKS ON *THE LOGIC OF SIGNIFICANCE AND CONTEXT*

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In this paper, I make two remarks on two points that Goddard and Routley make in Chapter 5 of [1]. The first remark is that the Brady disjunction, 'V', does have applications in ordinary discourse, an answer to doubts about this expressed by them on pp. 344-355. I show this by showing that each of four examples can be interpreted in a way which requires 'V' in its formalization. The second remark is that the need for a functionally complete significance logic seems very limited. I show this by examining each of the five arguments for a functionally complete system on pp. 348-351 of [1] and by showing how their weaker significance-complete systems will suffice, except in exceptional circumstances,

to express what they suppose a functionally complete system is needed for.

References

[1] Goddard, L. and Routley, F.R., *The Logic of Significance and Context*, Vol. 1, Scottish Academic Press, London, 1973.

NEW DIRECTIONS IN FOUNDATIONS OF MATHEMATICS

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The development of number system is traced historically initially. This results in further questioning which leads into anthropological and psychological considerations. This paper points out areas where new areas of foundational studies arise.

THE DIALECTICAL RULE OF STRENGTH

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It is misleading to say that some prisoners have escaped when one is in a position to say that they all have; to say that the book looked red when one is in a position to say that it was red; or to assert a disjunction when one is in a position to assert one of its disjuncts. In all these cases one is saying something *weaker* than what one is *in a position to say*. To account for this kind of misleadingness within C.L. Hamblin's formal theory of dialogue, it is necessary to analyse *weakness* as a syntactical relation. It is shown that weakness is different from immediate superimplication. What makes an utterance misleading is that it is weaker than another which the speaker is *in a position to say;* but whether an utterance is misleading cannot be immediately recognizable. Nonetheless, a rule of dialogue can be formulated governing misleadingness. It is then shown that this rule obviates the necessity for postulating either truth-value gaps or a relevance requirement on conditionals.

THING, NAME, NUMBER

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Logic started with things but now operates with names or quantities. This change is connected with a theory of knowledge. Modern Western theories of knowledge have been attempts to account for our apparent dealings with complex things, of various kinds, which were the 'material' of the propositions of traditional logic, but which - accepting Locke's basic account as true, as Hume and Kant did - we cannot, strictly speaking, be said to know. This

theory, which is not without its critics, accords with the existential assumptions of modern physics.

An unspoken axiom of traditional logic, springing from its theory of knowledge, was that logic is concerned with complex things and their relationships; an unspoken axiom of modern logic is that it is concerned with non-complex elements and ways of relating them. But in spite of these fundamental differences, the propositional expressions of the predicate calculus and the calculus of classes are based on the traditional four forms.

It is argued (1) that there are important differences between things, names and numbers and that statements in the traditional form cannot, as is supposed, be rendered equivalently in the modern forms and (2) that some of the logical paradoxes are due to failure to adhere to strict nominalism; and it is concluded that as it stands logic cannot avoid, but can only conceal, some treatment of things as complexes with relationships of their own.

MODEL THEORY VIA IMAGINARY UNIVERSES

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A technique for applying models of set theory to obtain results in model theory will be discussed. The methods are elaborated in detail in [1]. Example. Suppose one wishes to show that LQ, logic with the additional quantifier "there exists uncountably many", is compact (i.e. if every finite subset of a set Φ of sentences has a model, then Φ has a model).

A Assume every finite subset of Φ has a model.

B This fact is true in the real world V, and hence in a countable sub-model V_0 of V.

C Take an elementary extension W of V_0 with a non-standard finite integer β. In W it is true that, "the first β members of Φ have a model, *M*".

D Corresponding to *M* one easily constructs a model *M** of Φ in the real world V.

HENCE Φ has a model

Once the basic techniques are understood, the method can be applied to give short proofs of many new and "classical" theorems.

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[1] Hutchinson, J.E., "Model Theory via Set Theory", *Israel Journal of Mathematics*, forthcoming.

A TABLEAU SYSTEM FOR POSITIVE RELEVANT IMPLICATION.

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Following the nomenclature conventions of [1], we let TS be the tableau formulation of some arbitrary logic S. As well TV is classical propositional calculus and $TV_{\rightarrow}$ is its implication/negation fragment. A formula A is said to be a theorem of $TTV_{\rightarrow}$ iff a tableau can be constructed for $\overline{A}$ in which every branch closes once and only once. It was announced in [4] and Belnap and I show in [5] that the imposition of a relevance condition on this definition of theoremhood gives a system which precisely captures the implication/negation fragment of the Anderson/Belnap system R of relevant implication.

In this paper we show how this result may be extended so as to produce a tableau formulation of the positive fragment of R, TR_{+}. The system R_{+} is defined by the set of all the theorems of R that are formulated in the positive, i.e. negation-free, vocabulary. We include in this vocabulary *fusion* or *intensional conjunction* which can be defined as follows, $A \circ B =_{Df} \overline{A \rightarrow \overline{B}}$, but which we will take as primitive, and the *propositional constant* t - the conjunction of all logical truths.

The relationship of *componenthood* between formulas is defined *via* the following three clauses.

(i) Any formula A is a component of itself.
(ii) If a formula A is of the form B&C or B∘C, then B and C are components of A.
(iii) If A is a component of B and B is a component of C, then A is a component of C.

If B is a component of A we write this as A[B], while A[C/B] shall mean that the formula C is substituted for the component B of A in A. Further terminology is from [1], [3] and [5].

The rules for TR_{+} then are stated operationally as follows:

Conjunction

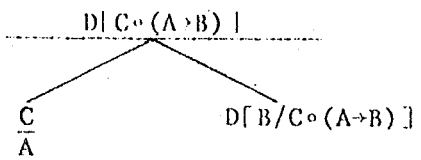
$$\frac{A\&B}{A \,\|\, B} \qquad \frac{\overline{A \vee B}}{\overline{A} \,\|\, \overline{B}}$$

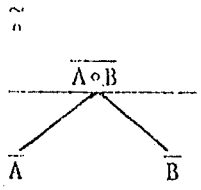
where A‖B means that *either* A or B *or both* can be assigned to successor nodes in the tableau.

Fusion

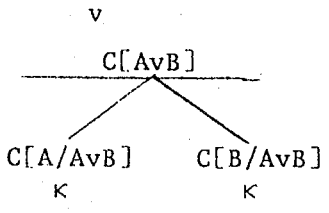
$$\frac{A \circ B}{\begin{array}{c} A \\ B \end{array}} \qquad \frac{\overline{A \rightarrow B}}{\begin{array}{c} A \\ \overline{B} \end{array}}$$

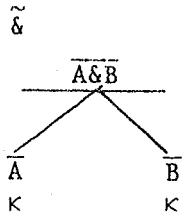
Relevant Implication





Disjunction





where we let μ be the origin branch that contains C[AvB] or $\overline{A\&B}$ and terminates with the node at which the rules ∨ or $\tilde{\&}$ are applied, such that κ is a sub-branch of μ that contains *at least* all the *unused* formulas that occur in μ.

Extensional Expansion

$$\mathrm{Ext}(\circ/\&)\qquad \frac{\begin{array}{c}A\\ \vdots\\ B_1\ \&\ldots\&\ B_n\end{array}}{(A\circ B_1)\ \&\ldots\&\ (A\circ B_n)}$$

where $n\geq 1$ and A is a positive formula.

Nested Extensional Expansion

$$\mathrm{Ext}[\circ/\&]\qquad \frac{C[A\circ(B_1\&\ldots\&B_n)]}{C[(A\circ B_1)\&\ldots\&(A\circ B_n)]}$$

where $n\geq 1$

As well we have two rules that govern the propositional constant t. The rule of t-*Introduction* (tI) simply allows t to be assigned to any empty node in a tableau, while the rule of t-*Elimination* (tE) when applied to some t appearing in a node in a tableau does not in any other way alter the tableau apart from rendering the t used. Finally we have the *Closure rule* (Cl) which when applied to two formulas A and $\overline{A}$ in some branch of a tableau means that no further construction need take place in that branch. Such a branch is said to be closed.

Theoremhood in TR_+ is defined as follows:

Definition 1. For all formulas A, $\vdash_{TR_+} A$ iff (i) A is constructed from the positive vocabulary, (ii) a tableau can be consructed for $\overline{A}$ in which every branch closes *once and only once* and (iii) *every* formula appearing at a node in the tableau has a rule applied to it at least once.

The proof of the equivalence of TR_+ and R_+ draws heavily on the deep insights into the proof-theoretic structure of R_+ that are to be found in the Gentzenization of R_+ given firstly by Dunn in [3] and later and independently, by Mints in [6] .Unfortunately the problem of extending our result in order to obtain a tableau system for all of R has till this date proved intractable.

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[1] Anderson, A.R. and Belnap N.D. Jr., *Entailment: The Logic of Relevance and Necessity*, Vol. 1, Princeton University Press, Princeton, New Jersey, 1975.

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SEMANTICAL INVESTIGATION OF ARRUDA-DA COSTA P SYSTEMS AND OF ADJACENT NONSUBSTITUTIONAL RELEVANT SYSTEMS.

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Systems in the vicinity of the P systems are logically important for several reasons. Firstly, there is the clear prospect that naive set theory, with an unrestricted comprehension axiom, erected with certain of these systems as underlying logic, will prove to be a non-trivial though inconsistent theory. Thus the systems offer important, if weak, bases for dialectical set theories. The prospect of a non-triviality demonstration extending earlier persistence arguments by adding world by world variation to the modelling that establishes the absolute consistency of extensional dialectical set theory - is enhanced through the lack of a rule of intersubstitutivity of coimplicants in P systems, and the resulting simplicity of the semantics of some of these systems. The failure of substitutivity is also the basis of a second reason for interest in these systems. There are good grounds for thinking that such semantically central notions as propositional identity and synonymy fail to satisfy intersubstitutivity rules. Thus the investigation of relevant systems in this vicinity -for propositional identity and synonymy surely preserve relevence- is a significant step in clarifying the logical features of these central notions. Furthermore most of the P systems, unlike the strong relevant systems studied by Anderson, Belnap and others, contains no principles whose logical correctness is in serious doubt - whereas the strong relevant systems contain many such principles.

Semantics for the system BH and one of its extensions GH illustrate the style of the semantical analyses given for the main P systems. BH has these postulates:- $A \to A$; $A, A \to B \Rightarrow B$; $A \to B, B \to C \Rightarrow A \to C$; $A \& B \to A$; $A \& B \to B$; $A, B \Rightarrow A \& B$; $A \to B, A \to C \Rightarrow A \to (B \& C)$; $A \to A \vee B$; $B \to A \vee B$; $A \to C, B \to C \Rightarrow A \vee B \to C$; $A \& (B \vee C) \to (A \& B) \vee (A \& C)$; $A \to B \Rightarrow \sim B \to \sim A$; $A \to \sim\sim A$; $\sim\sim A \to A$. GH results upon adding $A \vee \sim A$ to BH. A BH-model $\underset{\sim}{M}$ is a structure $\underset{\sim}{M} = \langle T, K, *, I\rangle$ where K is a set of situations, T (the actual situation) is an element of K, * is a reversal operation on K such that $a^{**} = a$ for $a \in K$, and I is an interpretation function from wffs of BH to $II = \{1,0\}$ which assigns as follows:- where A is a sentential parameter, $I(A,a)$ has, for each a in K, exactly one of the values in II, and where A is a wff of the form $B \to C$ and a is any world in K other than T, $I(A,a)$ has exactly one of the values of II. In short, for $a \neq T$, $I(B \to C,a)$ is assigned 1 or 0 arbitrarily, by the model. Interpretation function I is extended to all wffs of BH by the following inductive rules for every a in K and every wff A, B:-

$I(A \& B,a) = 1$ iff $I(A,a) = 1 = I(B,a)$; $I(A \vee B, a) = 1$ iff $I(A,a) = 1$ or $I(B,a) = 1$; $I(\sim A,a) = 1$ iff $I(A,a^*) \neq 1$; $I(A \to B,T) = 1$ iff $[A] \subseteq [B]$, i.e. iff for every a in K, if $I(A,a) = 1$ then $I(B,a) = 1$, since by definition $[A] = \{c \in K: I(A,c) = 1\}$. A wff A is true in $\underset{\sim}{M}$ iff $I(A,T) = 1$, and A is BH valid iff A is true in all BH-models. Semantics for GH result upon stipulating that the interpretation function satisfies the further condition: if $I(A, T) \neq 1$ then $I(\sim A, T) = 1$; such semantics thus relax usual inductive requirements. Strong adequacy theorems and their standard corollaries are forthcoming, e.g. a wff of HF is a theorem of HF iff it is HF-valid.

ON THE METAMATHEMATICS OF ORDINAL ARITHMETICS

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We begin with a discussion of the models of the first order theories $Th(\alpha,\leqslant)$, α an ordinal. Ehrenfeucht has analysed the well-ordered models of these theories and his approach may be extended quite naturally to account for the non-well-ordered (non-standard) models. Also from this approach one may deduce a basis for the Lindenbaum algebra of the theory

$\cap\{Th(\alpha,\leqslant)$: α an ordinal$\}$ [3].

In a similar way one may analyse models of the first order theories $Th(\alpha,\leqslant,+)$. In the language for these theories there is a formula whose interpretation for an element in a well-ordered model yields the Cantor normal form representation of that element. Hence many nice properties of the standard models are transferred to the non-standard models. For a subclass of the non-standard

models called the finitely formed models one may obtain results analogous to those described for ordinals by Ehrenfeucht [1]. An examination of some properties of these non-standard models highlights certain aspects of the classical treatment of standard structures given in [1]. Results may also be obtained for more extensive systems of ordinal functions [4].

In the last part of this paper we consider some general model theoretic constructions which were introduced by Feferman in [2] and later generalised by others. We relate this work to the work on non-standard models already discussed. We conclude with a brief survey of the literature dealing with applications of these particular model constructions.

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[3] E.A. Sonenberg, *On the elementary theory of inductive order*, Logic Paper No.22, Monash University, 1976.

[4] E.A. Wachs, *Models of well-orderings*, Ph.D. thesis, Monash University, 1976.

CURRENT RESEARCH IN RELEVANT LOGIC.

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Relevant logics have three roots: (1) Ackermann's system of *strenge Implikation* circa 1956, (2) work on weaker versions of the deduction theorem by Moh-Shaw-Kwei and by Church circa 1950, (3) the enormous extension of this work, with especial attention to its motivation via philosophical analysis of the notions of relevance and modality, by Anderson and Belnap. A treatise, *Entailment*, of which volume I has just appeared, gives full details concerning these roots, while also incorporating most of the published technical work in the area, both by Anderson and Belnap and by other authors.

The roots of relevant logic, however, have been dwelt on enough. The purpose of the present paper is to give the reader/listener some rudimentary information about the character of the tree which is springing from these roots. It is the author's conviction, not necessarily shared by all (or any) of his friends, that a new stage in the development of relevant logics has been

reached, which calls both for attention to different sorts of technical problems and even perhaps for significant rethinking of items of original philosophical and technical motivation.

Briefly, there now exist a *class* of logics with a great many family resemblances--deductive, semantic, algebraic--which, for lack of a better term, are called here relevant *logics*. The family is an extended one, and, like many such families, it is a little blurry at the edges what systems are to be taken as belonging to the class and which systems depart too far from fundamental insights to be listed as anything but distant cousins. On the minimal side is the Routley-Meyer system B+; maximal, perhaps, in the class, or perhaps minimally irrelevant, is the Sobocinski three-valued logic RM3 (which, however, became of interest as an extension of the Dunn-McCall system RM). Somewhere in the middle are the systems of principal interest in *Entailment*--namely, the system T of ticket entailment, and then, in order of increasing strength, the system E of entailment, R of relevant implication, and RM. (E, R, and T are due, of course, to Anderson and Belnap, though E is in fact identical in its stock of theorems to the earlier Ackermann system of *strenge Implikation*.)

Already to be taken for granted, though it was not always so, is that the principal members of the class of relevant logics are philosophically and technically stable in important ways, and that the principal *old problems* regarding these logics are now solved. (An exception is the question whether the Anderson-Belnap systems are decidable, though it is known that the weak system B+ and that the strong systems RM and RM3 are decidable, and also that there exist undecidable relevant logics in between, the latter also of a more or less natural sort.) The new problem of relevant logics is, principally, the question "What are their fertile uses?" How, for example, do concrete mathematical or scientific theories look when they are formulated in the context of a relevant logic? Aside from some rudimentary work on number theory, and some even more rudimentary work on set theory, little has been done in this area. Another question, central from the point of view of the old motivation, is how, and to what extent, intuitive notions about relevance will be reflected in serious theoretical application. Does it make sense, for example, to identify a Relevantist position in the foundations of mathematics, as one might identify an Intuitionist one? On such Large Questions, and some Small Ones, will the paper dwell.

A CONSISTENT ALTERNATIVE SET THEORY AND A NEW FOUNDATION FOR ARITHMETIC.

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This paper presents a consistent formal system ("J"), which can be regarded as an alternative set theory. A certain augmenta-

tion of the vocabulary of *any* orthodox set theory combined with acceptance of a strong form of the principle of the identity of indiscernibles, entails that one of the axioms of J, the analogue in J of the axiom of extensionality, must be preferred in place of the latter. Some intuitive reasons are given for preferring another of the axioms of J, the analogue of the axiom of comprehension of naive set theory to the latter. As well, this paper presents an axiomatization of arithmetic which is substantially different from that embodied in Peano postulates.

THE LOGIC OF PARADOX

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This paper falls into four sections.

It is argued that the failure of logicians to find generally accepted solutions to the logical paradoxes suggests that we try a new approach. Instead of trying to eliminate the paradoxes, suppose we accept them as "brute facts of life", i.e. some sentences are true, some false, and some just plain paradoxical.

An argument is given as to why we must, in fact, accept paradoxes. The Gödel incompleteness results arise because the language we are dealing with is not semantically closed (in the sense of Tarski). Thus if we are to come to grips with mathematical proof (in the naive sense), we must learn to handly semantically closed languages (such as English). It is well known that semantically closed languages give rise to paradoxes.

If we accept the above position, then clearly classical logic is going to prove inadequate for the purpose of formalization, since classically, a contradiction implies everything. Hence we must construct a new logic which does not suffer from the same problem. Moreover, the above considerations suggest a natural way of doing this. Following these suggestions, the logic of paradox LP is constructed and its properties considered.

Finally some open problems are discussed and tentative solutions examined.

COMBINATORY LOGICS AND LAMBDA CALCULI AS MODEL PROGRAMMING LANGUAGES.

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The classical systems of weak combinatory logic and lambda calculus are examples of replacement systems in the sense of Rosen. Replacement systems are primitive models of high level programming languages for which one can discuss models of problems such as; given that there are various alternative ways to execute a program, which ways are the most efficient?

A discussion of that problem for general replacement systems is given, and its applications to the classical system is briefly sketched. The main result is that in a system with a general computation step denoted $\rightarrow$, a subset $\rightarrow_o$ of $\rightarrow$ is optimal if it satisfies the following conditions.

a) If $a \rightarrow b$ for some b then for some c, $a \rightarrow_o c$.

b) There is a subset $\rightarrow_\delta$ of $\rightarrow$ such that, writing $a \rightarrow_\nu b$ in the case when $a \rightarrow b$ but neither $a \rightarrow_o b$ nor $a \rightarrow_\delta b$, the following conditions are satisfied.

(i) If $a \rightarrow_o b$ and $a \rightarrow_o c$ where $b \neq c$ then there is d such that $b \rightarrow_o d$, $c \rightarrow_o d$.

(ii) If $a \rightarrow b$, $b \rightarrow_o d$, then there is c such that $a \rightarrow_o c$ and $c \rightarrow_\delta^* d$, where * denotes some finite number, ≥ 0, of steps of the type indicated.

(iii) If $a \rightarrow b$, $a \rightarrow_o c$ then there is d such that $b \rightarrow_o d$, $c \rightarrow_\delta^* d$.

(iv) If $a \rightarrow_o b$, $a \rightarrow_\nu c$ then there is d such that $c \rightarrow^* d$, $b \rightarrow^* d$, where the steps of $c \rightarrow^* d$ are each either $\rightarrow_o$ of $\rightarrow_\delta$ and include at least one $\rightarrow_o$, and the steps of $b \rightarrow^* d$ include at most one step not in $\rightarrow_\delta$.

A LOGIC FOR INDEFINITE PROPOSITIONS

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Recently, R. Giles, [1], [2], [3] and [4], introduced an interpretation, using debates, of propositions that were true or false with definite probabilities. The resulting logic was Łukasiewicz's L^∞ He then extended this technique to give a semantics for propositions whose outcomes might have unknown probabilities. It is this logic we describe.

We consider propositions built using the connectives "&" (and), "v" (or) and "→" (implies) from a given set of prime propositions. Each prime proposition is associated with an experimental procedure that yields 0 or 1 ("yes" or "no") when a trial of it is performed. All the prime propositions are statistically independent of one another. Classical propositions always yield "yes" or always yield "no".

To gain a pragmatic meaning for propositions, ask what any speaker would intend by asserting a proposition. For example a speaker would assert P→Q in classical logic if he were prepared to assert Q whenever an opponent asserted P. When P and Q are non-classical propositions, take an assertion of P→Q to mean (as in L^∞) that the proponent will assert Q whenever an opponent challenges

by asserting P, and that these two assertions are then tried with each speaker paying his adversary if his own assertion is found false.

In general a speaker who asserts a proposition must engage in debate with any opponent until a situation (called a prime position) is reached that can be treated. Prime positions are like bets about the outcomes of trials. The definition of the rules of debate (more formally, a game) brings out ambiguities that coalesce in ordinary logic.

Each speaker will have a set of propositions that he accepts, in that he is prepared to assert them. A proposition that is acceptable to all speakers is called true. The true propositions are not closed under substitution. Let logical identities be those propositional forms that are true when arbitrary propositions are substituted for the prime propositions that occur in them. Logical identities are closed under *modus ponens* and contain a free lattice. Our main theorem gives a decision procedure for logical identities.

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AN ALTERNATIVE ACCOUNT OF BRINGING ABOUT (WITH A PINCH OF RELEVANCE).

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This paper gives an account of the logic of the bringing about operator D (read DA as 'a brings it about that A', or 'a makes it the case that A'), on which this operator does not distribute over conjunction. The motivation is that there seems to be a sense of the English paraphrases just given on which the implication 'D(p & q) ⊃ (Dp & Dq)' is not acceptable. If there is a cup on

the table, for example, and I put a saucer down on the table beside it, I certainly seem to bring it about that there is now both a cup and a saucer on the table, without thereby bringing it about that there is a cup on the table (since the cup was there all along, independently of any action of mine).

The failure of the principle in question makes a modal account of 'D', along such lines as are proposed by Ingmar Pörn and Douglas Walton, inappropriate. A four-valued semantical framework is suggested an an alternative, relative to whose matrices the following axioms and rules of proof give a sound, though not complete, axiomatization:

(0) A where A is an tautology
(1) $Dp \supset p$
(4) $(p \ \& \ Dq) \supset D(p \ \& \ q)$
(5) $D(p \ \& \ q) \supset (Dp \ v \ Dq)$

Uniform Substitution, Modus Ponens, and the rule:

(R $\supset$) from $A \supset B$ as a theorem, to infer $DA \supset DB$ as a theorem, provided that every propositional variable occurring in A also occurs in B.

The variable sharing restriction on this last rule is essential, in the sense that the rule stated without it would not preserve validity. As a result &/v-duality is interfered with, and while, for example we can prove '$Dp \supset D(p \ v \ q)$', we cannot prove (as intended, indeed) '$D(p \ \& \ q) \supset Dp$'. Some discussion is devoted to the question of whether such asymmetries are justified on any conception of bringing about.

Incompleteness and Inconsistency in Propositional Logic

John Grant

In this note I discuss two complementary notions that classical logic does not deal with adequately. Although I treat here only the propositional case, the matter certainly extends to predicate logic. Consider any system involving discrete items of information. This is a very broad notion and it may include the knowledge (or beliefs) of an individual, events described in a newspaper, or data stored in a computer. In standard propositional logic the assumption is made that every proposition is either true or false but not both. However an information system may very well have the property that the truth values of some propositions might be unknown. Also in such a system a proposition might be given (perhaps erroneously) both the values true and false.

Consider the case where the information system is the set of events described in a newspaper. Many events are not described at all while others are described in a contradictory fashion. Yet it should still be possible to make inferences and to gain knowledge by reading a newspaper. Belnap in [2] and [3] proposes a four-valued logic with truth-values True, False, Both, Neither to be used by an artificial information processor, a computer; he obtains very interesting connections with Scott's approximation lattices. Dunn deals with similar situations in several papers: in [4] for example he investigates the tautological entailment of Anderson and Belnap ([1] Chapter III) and gives a more natural presentation of Jeffrey's coupled trees. In [6] Gutiérrez discusses the problems of incompleteness (blanks) and inconsistency (knots) in the context of scientific practice and theory; he gives several examples of this phenomenon.

The purpose of this note is to give semantical and syntactical characterizations for propositional logic with possible incompleteness and inconsistency. Instead of using the phrases "A is true" and "A is false" I prefer to use the more neutral-sounding terminology "A is asserted" and "~A is asserted". Clearly there are four possibilities. In classical logic for every propositional variable either it or its negation (but not both) must be asserted. Thus no incompleteness or inconsistency is allowed. The other three cases arise by removing the restrictions mentioned above. I use the notation given in [5]; b-logic allows inconsistencies; n-logic allows incompleteness, and g-logic allows both possibilities. It seems worthwhile to look at these four cases using a single framework. I use the connectives ~ (not), $\wedge$ (and), $\vee$ (or) throughout.

To avoid any possible confusion I review first some terminology. $P = \{p_1, p_2, p_3, \ldots\}$ is the set of propositional variables; $L = \{p_1, \sim p_1, p_2, \sim p_2, \ldots\}$ is the set of literals.

I use l with subscripts for literals and call p and $\sim p$ complementary literals. The set of formulas F is the smallest class such that $P \subseteq F$ and if $A \in F$ and $B \in F$ then $\sim A \in F$, $(A \wedge B) \in F$, and $(A \vee B) \in F$. I use capital letters like A,B,C for formulas and leave out parentheses in the usual way. A conjunct is a disjunction of one or more literals, and a formula is in conjunctive normal form if it is a conjunction of one or more conjuncts. B is an immediate subformula of A if either $A = \sim B$ or $A = B \vee C$ or $A = C \vee B$ or $A = B \wedge C$ or $A = C \wedge B$. Then B is a subformula of A if there is a sequence of formulas $B_1, \ldots, B_n (n \geq 1)$ such that $B_1 = B$, $B_n = A$, and B_i is an immediate subformula of B_{i+1} for all i, $1 \leq i < n$.

The semantic approach uses the notion of valuation. A subset V of L is called an elementary valuation if for every pair of complementary literals $\{p_i, \sim p_i\}$ exactly one member of the pair is a member of V. This definition is modified for the other three cases: for elementary b-valuation replace "exactly" by "at least"; for elementary n-valuation replace "exactly" by "at most"; every subset of L is an elementary g-valuation. Every elementary valuation(resp. elementary b-valuation, elementary n-valuation, elementary g-valuation) V generates a valuation (resp. b-valuation, n-valuation, g-valuation) $\bar{V}$, where $\bar{V} \subseteq F$ as follows:

i) If A is a literal then $A \in \bar{V}$ iff $A \in V$.
ii) If $A = B \vee C$ then $A \in \bar{V}$ iff $B \in \bar{V}$ or $C \in \bar{V}$.
iii) If $A = B \wedge C$ then $A \in \bar{V}$ iff $B \in \bar{V}$ and $C \in \bar{V}$.
iv) If $A = \sim B$ then
    a) if $B = \sim C$ then $A \in \bar{V}$ iff $C \in \bar{V}$,
    b) if $B = C \wedge D$ then $A \in \bar{V}$ iff $\sim C \in \bar{V}$ or $\sim D \in \bar{V}$,
    c) if $B = C \vee D$ then $A \in \bar{V}$ iff $\sim C \in \bar{V}$ and $\sim D \in \bar{V}$.

I write $A_1, \ldots, A_n \vDash B$ if for every valuation $\bar{V}$, whenever $\{A_1, \ldots, A_n\} \subseteq \bar{V}$ then $B \in \bar{V}$. Let $i \in \{b,n,g\}$. $\vDash_i$ is defined by changing "valuation" above to "i-valuation". If $A \in \bar{V}$ for every i-valuation $\bar{V}$ then A is an i-tautology; if $A \notin \bar{V}$ for any i-valuation $\bar{V}$ then A is an i-contradiction. Clearly the b-tautologies are just the usual tautologies; the n-contradictions are the usual contradictions; there are no b-contradictions and no n-tautologies. Define A and B to be i-equivalent if for every i-valuation $\bar{V}$, $A \in \bar{V}$ iff $B \in \bar{V}$. The formulas $p_1 \wedge \sim p_1$ and $p_2 \wedge \sim p_2$ are n-equivalent but not b-equivalent; the formulas $p_1 \vee \sim p_1$ and $p_2 \vee \sim p_2$ are b-equivalent but not n-equivalent; the formulas $p_1 \vee \sim p_1$ and $(p_1 \vee \sim p_1) \vee (p_2 \wedge \sim p_2)$ are both b-equivalent and n-equivalent but not g-equivalent.

The syntactical approach uses a deductive system with fairly straightforward rules so that any of the logics might be obtained by simple changes. The explanation of the use of the rules follows their presentation. There are 11 rules each with parts a) and b).

1 a) $\dfrac{\sim\sim A}{A}$ b) $\dfrac{A}{\sim\sim A}$

2 a) $\dfrac{\sim(A\wedge B)}{\sim A\vee\sim B}$ b) $\dfrac{\sim A\vee\sim B}{\sim(A\wedge B)}$

3 a) $\dfrac{\sim(A\vee B)}{\sim A\wedge\sim B}$ b) $\dfrac{\sim A\wedge\sim B}{\sim(A\vee B)}$

4 a) $\dfrac{A\wedge B}{B\wedge A}$ b) $\dfrac{A\vee B}{B\vee A}$

5 a) $\dfrac{A\wedge(B\wedge C)}{(A\wedge B)\wedge C}$ b) $\dfrac{(A\wedge B)\wedge C}{A\wedge(B\wedge C)}$

6 a) $\dfrac{A\vee(B\vee C)}{(A\vee B)\vee C}$ b) $\dfrac{(A\vee B)\vee C}{A\vee(B\vee C)}$

7 a) $\dfrac{A\wedge(B\vee C)}{(A\wedge B)\vee(A\wedge C)}$ b) $\dfrac{(A\wedge B)\vee(A\wedge C)}{A\wedge(B\vee C)}$

8 a) $\dfrac{A\vee(B\wedge C)}{(A\vee B)\wedge(A\vee C)}$ b) $\dfrac{(A\vee B)\wedge(A\vee C)}{A\vee(B\wedge C)}$

9 a) $\dfrac{A\vee A}{A}$ b) $\dfrac{A}{A\vee B}$

10a) $\dfrac{A\wedge B}{A}$ b) $\dfrac{A,B}{A\wedge B}$

11a) $\dfrac{}{A\vee\sim A}$ b) $\dfrac{A\vee B,\sim A\vee C}{B\vee C}$

All the rules except for 10b), 11a), and 11b) have the form $\dfrac{Q}{R}$ where Q and R are formulas. These rules are used in the following way with one proviso: if Q is a subformula of T and T' is obtained from T by one substitution of R for Q then T' is said to be immediately deducible from any set of formulas $X \supseteq \{T\}$. The proviso is that in the case of rules 9b) and 10a) Q is not allowed to be a subformula of a formula Q' where ~Q' is a subformula of T. Finally rules 10b), 11a), and 11b) are used as follows: the formula A∧B is immediately deducible from any set of formulas $Y \supseteq \{A,B\}$; the formula

$A \vee \sim A$ is immediately deducible from any set of formulas (including ϕ); and the formula $B \vee C$ is immediately deducible from any set of formulas $Z \supseteq \{A \vee B, \sim A \vee C\}$.

Given a set of formulas S a proof of B from S is a finite sequence of formulas $B_1, \ldots, B_k$ where $B_k = B$ and each B_i is immediately deducible from $S \cup \{B_1, \ldots, B_{i-1}\}$. If there is a proof of B from S then I write $S \vdash B$; if $S = \{A_1, \ldots, A_n\}$ I may write $A_1, \ldots, A_n \vdash B$. $\vdash$ is simply the deducibility relation of ordinary logic. Note that using rules 1-9 it can be shown that for every formula A there is a formula B in conjunctive normal form containing the same set of propositional variables such that $A \vdash B$ and $B \vdash A$. Next, let A' be a conjunctive normal form of A obtainable from A by rules 1-10 and let B' be a conjunctive normal form of B obtainable from B by rules 1-10. Then $A \vdash B$ iff $A' \vdash B'$. Thus it suffices to consider only proofs of the form $A \vdash B$ where A is in conjunctive normal form, B is a conjunct, and there are no applications of rules 1,2,3. Now with a proviso about reordering literals and eliminating parentheses it can be seen that only rules 7-11 are needed. But since the final result is to be a conjunct rules 7 and 8 need not be used. So finally only rules 9, 10, and 11 are needed. Furthermore, in the use of these rules A,B,C may be restricted to be conjuncts and for rule 11 it suffices to consider the case where A is a propositional variable. Actually rule 11a) is needed only in the proof of a tautology.

Completeness. $A \vdash B$ iff $A \models B$.
Proof: Assume that $A \models B$. Then $\sim A \vee B$ is a tautology. By rules 9b), 11b), and 9a) $A, \sim A \vee B \vdash B$. Thus it suffices to show that if C is a conjunct and a tautology then $\vdash C$. But this is the case exactly if $C = l_1 \vee l_2 \vee \ldots \vee l_i$ where at least two of the literals in C are complementary. The result then follows by rules 11a) and 9b).

The deductive system given above has the nice property that by omitting rule 11b) b-logic is obtained, by omitting rule 11a) n-logic is obtained, and by omitting both 11a) and 11b) g-logic is obtained. Then $\vdash_i$ is defined as $\vdash$ except for the omission in a proof of the i appropriate rule(s).

Completeness for b-logic. $A \vdash_b B$ iff $A \models_b B$.
Proof: Assume that $A \models_b B$. If B is a b-tautology then the result follows from the proof of Completeness. So assume that B is not a b-tautology. It suffices to consider the case where $C_1 \wedge \ldots \wedge C_k \models_b l_1 \vee \ldots \vee l_i$, the C_j being conjuncts and the l_j literals. Since B is not a tautology there is no complementary

pair of literals in the set $S = \{l_1,\ldots,l_i\}$. By the definition of $\models_b$ whenever $\{C_1,\ldots,C_k\} \subseteq \bar{V}$ some member of $S \in \bar{V}$ for any b-valuation $\bar{V}$. This means that there is at least one C_j such that each literal of C_j is a member of S. Now apply rules 10a) and 9b).

Completeness for n-logic. $A \vdash_n B$ iff $A \models_n B$.
Proof: Assume that $A \models_n B$. It suffices to consider the case where $C_1 \wedge \ldots \wedge C_k \models_n l_1 \vee \ldots \vee l_i$, the C_j being conjuncts and the l_j literals. Proceed by induction on k. If $k = 1$ then by the definition of n-valuation C_1 may contain only literals from the set $S = \{l_1,\ldots,l_i\}$. Now apply rule 9b). Next assume the result for k-1 conjuncts. If C_k contains only literals from the set S then apply rules 10a) and 9b). So it may be assumed that C_k has the form $l_1 \vee \ldots \vee l_j \vee l_{i+1} \ldots \vee l_{i+m}$ where $\{l_1,\ldots,l_j\} \subseteq S$, $T = \{l_{i+1},\ldots,l_{i+m}\}$ and $S \cap T = \emptyset$. Write l' for the complement of a literal l. By using the definition of $\models_n$ obtain $C_1 \wedge \ldots \wedge C_{k-1} \models_n l_1 \vee \ldots \vee l_i \vee l'$ for each $l \in T$. Thus by the inductive hypothesis, $C_1 \wedge \ldots \wedge C_{k-1} \vdash_n l_1 \vee \ldots \vee l_i \vee l'$ for each $l \in T$. Now apply rules 10a), 9b), 9a), as well as 11b) m times to obtain $C_1 \wedge \ldots \wedge C_k \vdash_n l_1 \vee \ldots \vee l_i$.

The Interpolation Theorem has a nice proof in the deductive system given above.

Interpolation. Assume that $A \vdash B$. Then at least one of the following three conditions holds:
1) A is a contradiction.
2) B is a tautology.
3) There is a formula C such that $A \vdash C$ and $C \vdash B$ and every propositional variable of C occurs both in A and in B.

Proof: First, $p \wedge \sim p \vdash q$ is an example of 1) and $p \vdash q \vee \sim q$ is an example of 2). Next assume that neither 1) nor 2) holds. Then by ignoring rule 11a) observe that there is only one rule, 9b), which allows the introduction of new variables in a proof. There are two rules which allow the elimination of variables in a proof. But any proof can be rewritten, if necessary, in such a way that any application of rules 10a) and 11b) occurs before any application of rule 9b). C can be therefore chosen as one of the formulas in the proof of $A \vdash B$.

The Interpolation Theorems for b-logic, n-logic, and g-logic are proved similarly. For b-logic omit condition 1); for n-logic omit condition 2); and for g-logic omit both conditions 1) and 2) to obtain the Perfect Interpolation Theorem ([1] Page 161).

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Some Entailments Not Strict Implications

We all know what entailment, or, as Moore called it, the converse of deducibility, is. It is that relation which, if it holds between the premise(s) and the conclusion of an argument, makes the argument valid. Our intuitions about entailment are a prime source of evidence for logical theory.

Strict implication, by contrast, is a philosopher's notion. We may define it as follows: a set of premises strictly implies a conclusion if and only if there is no possible world in which the premises are all true and the conclusion false.

We may ask whether these two notions coincide - that is, whether a set of premises entails a conclusion if and only if it strictly implies it.

A number of philosophers in Pittsburgh believe that there are sets of premises which strictly imply conclusions which they do not entail. Thus the premises 'Snow is white' and 'Snow is black' strictly imply the conclusion 'George Gaylord Simpson is a whale', for, as there is no possible world in which the premises are both true, there is certainly no possible world in which the premises are true and the conclusion false. It has been claimed, however, that the absence of any connection between the subject matters of the premises and the conclusion prevents them from entailing it. I do not wish to commit myself on this question, however.

I do not know if these philosophers have addressed themselves to the question of whether, if a set of premises entails a conclusion, it will also strictly imply it. This much is clear, though not particularly relevant: if one can prove formally (say by using the Anderson-Belnap system E of entailment) that a set of premises entails a conclusion on their conception of entailment, then one can also prove formally that it strictly implies the conclusion. This is not particularly relevant: if, say in one of the standard systems of modal logic, one can prove formally that a certain proposition is true, then one can also prove formally that it is necessarily true. But not all true propositions are necessarily true.

Let us define S as the set of all human beings who have ever lived, are alive now, or will ever live. Set S is surely well enough defined for us to reason about it (we may suppose any vagueness about just what is a human being cleared up by some arbitrary stipulation - for what follows it will not matter what stipulation). Now, it seems to me that one may infer, validly, from the premise that at some time no member of S was in existence, that at that time there were no human beings. That is, the (unit set of the) premise 'At one time no member of S was in existence' entails the conclusion 'At one time no human being existed'. This premise does not, however, strictly imply the conclusion. For surely there is a possible world in which human beings exist at every moment of time (thus falsifying the conclusion), but in whose history there is a moment such that each human being existing in that world at that moment is distinct from any human being ever existing in the actual world, making the premise true.

Many other counterexamples may be obtained along similar lines. The existence of such counterexamples is somewhat surprising, I think, because many philosophers have introduced the notions of 'possible world' and 'strict implication' precisely in order to explicate such notions as that of 'entailment'. Perhaps the simplest such example, and one which may help suggest an explanation for our "phenomenon", is this: if we are told that a certain proposition is true in the real world, we may infer that it is true. But of course there are possible worlds at which propositions true at the real (actual) world are false.

I would like to conclude by sketching the outlines of what I take to be the explanation of these examples. The importance of valid arguments is that their premises, if true, "guarantee" the truth of their conclusions. We want to say that this guarantee amounts amounts to more than just saying that if the premises are true, the conclusion is too, with the 'if' construed as material implication. One attractive approach to explicating this "more" is to say that the material conditional, if the premises are true the conclusion is too, not only holds as a matter of fact, but would hold throughout a wide range of circumstances. This approach produces a formal logic of entailment which coincides with the formal logic of strict implication. Now, for most of the propositions we concern ourselves with, we can take the range of alternative circumstances relevant to questions of entailment to be the set of possible worlds. When we explicitly refer to possible worlds, however, a possible world is no longer "big enough" to serve as an alternative. It seems that we must consider, at the very least, alternative systems of possible worlds, each system with its own "actual" world, if we are to maintain the range of alternatives approach.

A final footnote: I often catch myself using modal terminology or counterfactual conditionals in discussing possible-worlds semantics for modal logic or counterfactuals. What semantic account am I to give of the language in which I do semantics?[1]

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[1]I would like to thank Alvin Plantinga for publishing an argument that inspired my first counterexample, and Joseph Camp and David Liden for moral and material support.

S5--The Poor Man's Connexive Implication

The recent explosion of interest in connexive implication, ignited by Angell and McCall, has left in its wake certain problems. Chief among them is the fact that, although almost everybody these days would like to have his own connexive implication, the price that has been put on extant systems is too high for the average pocketbook. Long lists of new axiom sets must be bought, and unfamiliar theses must be purchased in these hard times when, frankly, consumer resistance has depressed the market even in some of the classical tautologies. When people won't buy paradoxes, it would seem fruitless to try to sell them theorems that even classical logicians--notorious spendthrifts, they--wouldn't touch, but this is precisely what the partisans of connexive implication have been attempting. Try on $A \nrightarrow \bar{A}$, they tell us, and they call it 'Aristotle' to interest purchasers of antiques. No need to pay cash, goes the pitch, for you may trade in your old models of $A\&B \nrightarrow A$ for down payment.

But, all things considered, $A\&B \nrightarrow A$ has given faithful service over the years, and logic, having outgrown Aristotle, is just not ready yet for 'Aristotle'. What is needed, rather, is just a connexive implication that everybody can afford, which can be put together cheaply from one or another old system that is lying around unused in the basement. S5, having been found unfit to reason with on many grounds--it's incoherent, to lapse into technical talk--is just such a system. With a little paint and a slight tune-up, your old S5 will cruise connexively down the rails of inference with all the abandon of a new high-priced 1975 McCall. Indeed, follow this guide and your 1932 Lewis will outperform the McCall, which has been known to stall at higher degrees. All that is required is a few minutes of your time.

I

The idée maîtresse of connexive implication, according to McCall 1975, is that no proposition should entail or be entailed by its own negation. ['Aristotle']. This idée found its first formal expression in matrix form in Angell 1962 and was reduced to syntactical form in McCall 1966. Unfortunately, as McCall 1975 points out, in its original form connexive implication resisted vehemently the notion that & is idempotent, as Routley and Montgomery 1968 pointed out in detail. True, $A\&A \nleftrightarrow A$ rarely figures in such arguments as might be counted elegant, but such is the force of custom among logicians that even in the name of Aristotle its defenders couldn't be moved.

Accordingly, McCall 1975 presents another system of connexive implication. We shall call it McC. McCall's underlying, and pretty, semantical picture may be presented as follows. Suppose we have a collection C of propositions, closed under the truth-functional operations &, V, -. Classically these form a Boolean algebra, on which a relation $<$, corresponding to classical entailment, may be defined by setting $a<b$ iff $a\&b$ and a are the same proposition. (Entailment is not to be here confused

with material implication: the former is a relation on C, while the lattee is an operation on C definable by $a \supset b = \bar{a} \vee b$, for all $a,b \in C$.) Moreover, since C is supposed to be composed of propositions, presumably exactly half of them are true; we let T be the set of the true propositions (the truth filter, as Belnap would put it), and we note that usual truth-table considerations suffice to make the truth filter a maximal filter in the Boolean algebra C, ordered by $\leq$. The complement $F = C-T$ (the false ideal, in Belnap's lingo) is likewise a maximal ideal in C.

Now now is a relation $\Rightarrow$ of connexive entailment to be defined on C? Sticking to the idee maîtresse, and continuing the temporary policy of using 'entailment' for a relation and 'implication' for a corresponding operation, McCall suggest that we can get connexive entailment out of classical entailment on the following specification:

(1) $a \Rightarrow b$ if and only if both (i) $a \leq b$ and (ii) $a \in T$ iff $b \in T$.

Thus connexive entailment is a tighter relation than classical entailment, and it is tighter on the point that falsehoods do not connexively entail truths. Otherwise connexive entailment and classical entailment on C coincide; falsehoods $\Rightarrow$ falsehoods iff they $\leq$ them, and truths $\Rightarrow$ truths iff they $\leq$ them. In the remaining case, truths neither connexively nor classical entail falsehoods, we hope.

Let us look again at (1). Suppose we were trying to define $a \Rightarrow b$ on some theory of propositions, classically based, which defined an operation corresponding to classical entailment. Obviously this operation would not be material implication, which is too weak. The obvious next choice is strict implication. That requires the ability to define a necessity operator N on C, whence strict implication $\dashv$ may be defined as an operation by setting

(2) $a \dashv b =_{df} N(a \supset b)$.

We suppose that our theory of necessity comes from some modal logic--S5, say.

According to (1) and given the musings just above, $a \Rightarrow b$ should hold iff (i) $a \dashv b$ is true and (ii) $a \equiv b$ is true, where $\equiv$ is just material equivalence. But (i) and (ii) together amount, on the usual truth condition, to $(a \dashv b) \& (a \equiv b)$ is true; i. e., to $(a \dashv b) \& (a \equiv b) \in T$, for T is the truth-filter in C.

At this point, however, we need no longer be so sticky about $\Rightarrow$ being only a relation (which limits us, on the syntactical side, to a consideration of just the first-degree formulas of McC, in which there is no nesting of $\Rightarrow$'s.) An S5 theory of necessity, after all, permits such nesting, and so we may just come right out and define connexive implication $\rightarrow$ by

(3) $a \rightarrow b =_{df} (a \dashv b) \& (a \equiv b)$.

(3) is a definition that makes sense for S5 as for its corresponding algebras; the connection with $\Rightarrow$ in C is that

(4) $a \Rightarrow b$ iff $a \rightarrow b \in T$.

II

We have now motivated our poor man's connexive implication, and we

must show that, as far as the connexive entailment (as we called it) of McC goes, the systems McC and PoorMan exactly coincide, PoorMan being just S5 enriched with the definition (3).

PoorMan is assumed known; it is based on a language SL with sentential variables p, q, r, etc., and with formulas A, B, etc. built up as usual from &, ∨, -, N, with ⊣ and ↠ defined by (2) and (3) above, and ⊃ and ≡ defined by the usual truth-functional definitions. Its set of theorems is that of S5. In primitive notation, a zero-degree formula (zdf) of SL is one which contains no occurances of N. First-degree formulas are characterized recursively (fdf) as follows: (i) if A is a zdf, A is an fdf and NA is an fdf; (ii) if A, B are fdf, $\overline{A}$, A&B, A∨B are fdf; (iii) nothing is an fdf except in virtue of (i)-(ii). We call the set of all first-degree formulas of SL the language SL_1, and we introduce for SL_1 (only) the additional definition

(5) $a \Rightarrow b =_{df} a \twoheadrightarrow b$.

Note that (5) only makes sense for SL_1. (Strictly speaking the letters in it, and previous definitions when taken syntactically, should be Roman capitals A, B, but we leave it to the reader to make trivial shifts from syntactic to algebraic gears and conversely.)

McC is based on a language ML_1, with connectives &, ∨, - and a partial connective ⇒. Its sentential variables and zero degree formulas are those of SL; otherwise, (i) if A is a zdf of ML_1, A is an fdf of ML_1; (ii) if A and B are zdf of ML_1, A⇒B is a fdf of ML_1; (iii) if A and B are fdf of ML_1, so are $\overline{A}$, A&B, A∨B fdf of ML_1. Thus, we note, ML_1 coincides with the set of its fdf. Introduced by truth-functional definition are ⊃ and ≡ as usual. Moreover we define in ML_1

(6) $t =_{df} p \supset p$, and

(7) $NA =_{df} t \twoheadrightarrow a$.

We define the set of theorems of McC following McCall 1975, making use of the motivating remarks of I, with terminological variations drawn from Belnap 1967. A connexive algebra is a pair ⟨C,T⟩, where C is a Boolean algebra and T is a maximal filter in C. A valuation of ML_1 in a connexive algebra ⟨C,T⟩ is a function v which assigns an element v(p) of C to each sentential variable p of ML_1, and then recursively to all zdf by setting $v(A\&B) = v(A) \wedge v(B)$, $v(A \vee B) = v(a) \vee v(B)$, $v(\overline{A}) = -v(A)$. An interpretation I_v determined by a valuation v in ⟨C,T⟩ assigns one of T, F to every formula of ML_1 on the following plan:

(i) If A is a zdf, I(A) = T iff $v(A) \in T$.

(ii) If A, B are zdf, I(A⇒B)=T iff both (a) v(A) = v(A&B) and (b) I(A) = I(B).

(iii) If A and B are fdf, I(A&B), I(A∨B), and I(A) are as determined by truth tables. ($I(\overline{A})$ = T iff I(A) = F, etc.)

A formula A is true on an interpretation I in a connexive algebra ⟨C,T⟩ iff I(A) = T, and is otherwise false on I. A is McC-valid iff A is true on all interpretations in all connexive algebras. The set of theorems of McC is exactly the set of McC-valid formulas of ML_1.

Note that in giving our recursive characterization of truth on

interpretation we were only echoing the motivating remarks; in particular, the valuation v may be thought of merely as assigning propositions to zdf, whence the truth-definition built into I is straightforward; in particular, the truth-condition (ii) precisely copies the informal remarks under (1) characterizing when it is that a→b holds.

III

The definitions (3), (5), (6), and (7) permit straight-forward translation from SL_1 to ML_1 and vice versa. We note, too, that definitions (5) and (7) are well-grounded in the following sense:

Ad (5), (3). $A \Rightarrow B \equiv (t \Rightarrow .A \supset B) \& (A \equiv B)$ is a theorem of McC, the left side being well-formed iff the right side is well formed.

Ad (7). $NA \equiv (t \rightarrow A) \& (t \equiv A)$ is a theorem of S5.

Since provable material equivalence is the natural congruence for both systems, our observations assure that a translation from, say, S5 to McC and back again to S5 returns us, essentially, to our starting point; accordingly, if we can show that theoremhood is preserved on translation in both directions, we shall show that McC and the first-degree fragment of S5 amount, essentially, to the same thing, justifying our initial claims for PoorMan.

As a start, let us see how some of the special theses of McCall 1975 turn out in PoorMan.

Ad Aristotle. Define AoB as $\overline{A \rightarrow \overline{B}}$ in S5 and as $\overline{A \Rightarrow \overline{B}}$ in McC. 'Aristotle' says that A is self-consistent, whatever A may be and McCall 1975 shows AoA McC-valid. Applying definitions, we see that

(8) $aob =_{df} \Diamond(a \& b) \vee . a \equiv b$

will do as an S5 definition of connexive consistency. Evidently, however, AoA then is a theorem of S5, since the second disjunct of (8) becomes A≡A, which is on its own hook a theorem.

Ad Boethius. The thesis of McCall 1975 to be verified in S5 is $A \rightarrow B . \supset . AoB$. But on definitions the antecedent has A≡B as a conjunct and the consequent has it as a disjunct, making it trivially a theorem. (McCall 1975, incidentally, would like Boethius to hold with ⇒ as the main connective, though McC doesn't satisfy the condition inasmuch as Boethius so expressed isn't an fdf; I think the desire misplaced, since what is want is that the inference from A→B to AoB should go through, but note for the record that in the stronger form Boethius is S5-invalid.)

And now lets examine some of the non-theorems.

Ad &E. We wish to show that A&B→A isn't a theorem of S5. Of course it's not, since (3) requires that antecedent and consequent of → materially equivalent.

Ad Mingle-fallacies. Let A be a truth-functional contradiction and B be a truth-functional tautology. Then A→B isn't a theorem of S5, for contradictions and tautologies aren't materially equivalent.

Well, that's enough fun. We simply observe that, in PoorMan, the non-classical character of connexive implication disappears, except insofar as it is non-classical to construe implication as a crazy mix of strict implication and material equivalence. The last wasn't my idea, though for all I know perhaps Aristotle really thought, on suitable explication, that A really implies B iff A implies B in all possible worlds and is in particular equivalent to B in this one, and that a sufficient condition for the consistency of A and B is that they should have the same truth-value, in this world. Anyway, it's pleasant to do philosophy in a system in which most everything is consistent with almost everything else, so that--at least-- the author is henceforth going to do metaphysics exlcusively in McC; as, he observe, most of his metaphysical friends seem already to be doing.

IV

We wish in this section to show that all theorems of McC are on translation theorems of S5. McCall 1975 axiomatizes McC, and I note that on translation each of his axioms is a theorem of S5 and that the rule (detachment for horseshoe) is a rule of S5. This ends the business of the section.

V

We wish in this section to show that all first-degree theorems of S5 are on translation theorems of McC. We proceed by contraposition, assuming that A is invalid in McC and showing that its S5-translate is S5-invalid. Since A is invalid, there is a valuation v and an associated interpretation I in a connexive algebra $\langle C,T\rangle$ such that A is false on I. Our plan is to add an operation N to C and to use our valuation v to construct an s5-valuation v' with the following properties:

(i) v' is defined on all formulas of Sl and has values in C.
(ii) for all translates B of zdf B, v'(B) = v(B)
(iii) for all translates B of fdf B, v'(B)ϵT iff I(B) = T.
(iv) for all theorems B of S5, v'(B)ϵT.

We begin by defining N on $\langle C,T\rangle$. Let 1 be the unit and 0 be the zero of the Boolean algebra C. We define N now as follows:

(a) N1 = 1; otherwise Na = 0.

We now define our valuation v' on SL by setting v'(p) = v(p) for sentential variables, and otherwise letting v'(A&B) = v'(A)&v'(B), v'(A∨B) = v'(A)∨v'(B), v'($\overline{A}$) = -v'(A), and v'NA = Nv'(A).

We have turned C into what, in the trade, is called a Henle matrix, and we leave it to the reader to check that the condition (iv) has accordingly been satisfied. We note too that our definition of v' has automatically satisfied conditions (i) and (ii), so that only condition (iii) remains.

We prove (iii) by induction on the length of B. It is immediate from (ii) and the definition of I in section II if B is a zdf, and it is trivial on inductive hypothesis and the maximal filterhood of T if B is of the form C&D, C∨D, or -C.

In the remaining case B is of the form C⇒D, where C and D are zdf. Let b be v'(B), c be v'(C), and d be v'(D). By definitions, $b = N(c \supset d) \& (c \equiv d)$. We need to show $b \in T$ iff B is true on the McC interpretation I; i. e., by (ii) of section II, iff both (a) $c = c \& d$ and (b) $I(C) = I(D)$. Since T is a filter, $b \in T$ iff both (a') $N(c \supset d) \in T$ and (b') $c \equiv d \in T$. Obviously it suffices to show that (a) holds iff (a') holds and that (b) holds iff (b') holds.

First, $c = c \& d$ iff $1 = c \vee \bar{c} = \bar{c} \vee d = c \supset d$ by Boolean properties, iff $1 = N1 = N(c \supset d) \in T$, by our Henle definition of N; so (a) holds iff (a') holds. Second, $I(C) = I(D)$ by definition of I, iff both $c \in T$ and $d \in T$ or else both $\bar{c} \in T$ and $\bar{d} \in T$; iff $c \& d \vee \bar{c} \& \bar{d} \in T$ by maximal filter properties; iff $c \equiv d \in T$ by definition. This shows that (b) holds iff (b') holds, completing the inductive argument for (iii). So (iii)-(iv) have been established, which as we saw above is enough to show that if A is a first-degree theorem of S5, then A is a theorem of McC. This ends the proof that McC is exactly contained in PoorMan. Since, however, PoorMan defines ⇒ not only at the first degree but on all formulas, the claim that your Lewis 1932 will outperform the McC on rough terrain while getting the same good mileage on the freeway must be taken as established.

VI

Now that we know that McC can be viewed as a strange way of doing an old logic as well as a new way of doing a strange logic, the question of a reformulation of the rather considerable number of axioms of McCall 1975 comes up. The underlying point, of course, is that we are merely seeking a smooth formalization of the first-degree fragment of S5, dressed up to look as though connexive implication was a reasonable primitive for it. (We have preferred S5 as our candidate for PoorMan--chiefly because we can't think of anything else that S5 is good for--but we note that inasmuch as it's only the first-degree fragment that we're interested in, and since the deeper problems of iteration of modalities that worried Lewis and differentiate systems don't arise at this level, any Lewis system whose first-degree fragment is the same as S5--e. g., S4-- would be an equally acceptable candidate for present purposes; indeed, the considerations that favor one way of iterating connexive modalities not being presently clear, perhaps some other system would be superior.)

First, what's a good way of axiomatizing the first-degree fragment of S5? On this occasion let's take just -, ⊃, N as primitive, defining other connectives familiarly and still calling the first-degree language SL_1. As axioms of our first-degree system $S5_1$, we take all formulas of SL_1 of the following kinds:

A1. NA, where A is a truth-table tautology
A2. A, where A is a truth table tautology
A3. NA⊃A
A4. N(A⊃B)⊃.NA⊃NB.
As sole rule, we take modus ponens for ⊃; for short, ⊃E.

Evidently everything provable in $S5_1$ is S5-valid, and we turn to the converse. Let C be a theorem candidate.[1] In view of A2 and ⊃E, we can use classical logic, so we may assume that C is in conjunctive normal form, i. e., that C is of the corm $C_1\&\ldots\&C_n$; that each of the C_i is of the form $C_{i_1}\vee\ldots\vee C_{i_n}$, and that each of the C_{i_j} is of the form p, -p, NA, or -NA, where p is a sentential variable and A is a zdf; let us call a C_i in this form a basic disjunction. (That A must be a zdf is clear from the requirement that C be a fdf, in our characterization of basic disjunctions.) On truth-functional considerations evidently we can prove C iff we can prove all the C_i in $S5_1$. So it suffices to assume that C is a basic disjunction and a theorem of S5; we show that C is a theorem of $S5_1$.

In fact, C is of the form, without loss of generality, $NA_1\&\ldots\&NA_m\supset.NB_1\vee\ldots\vee NB_n\vee D$, where each of the A_i, B_i, and D are zdf. The NB_j do not interact with each other or with D, as the reader may use his favorite metalogical tool (e. g., Kripke semantics) to discover. So C is a theorem of S5 iff, for some j, $1\leq j\leq n$, (a) $NA_1\&\ldots\&NA_m\supset.NB_j$ or (b) $NA_1\&\ldots\&NA_m\supset D$ is a theorem. Truth-table considerations, and elementary syntactical Lewis properties, show that these will hold respectively, iff either $A\supset B_j$ or $A\supset D$ is a truth-table tautology, where A is the conjunction of the A_i. But if $A\supset D$ is tautologous, it is a theorem of $S5_1$ by A2, whence repeated use of A3 yields a proof of (b) in $S5_1$ by classically sanctioned moves. Similarly if $A\supset B_j$ is tautologous, so by A1 is $N(A\supset B_j)$, whence repeated use of A4 does the trick. Degenerate cases (when, e. g., there are no A_i, or no B_j, or no D) are handled in like manner, ending the proof of the adequacy of our axiomatization of $S5_1$. Or of $S4_1$. Or $S3_1$. Or M_1. Or $S2_1$, Or $S1_1$, Or $SO.5_1$, for that matter, since for at least that far the above axioms and rules, for formulas of SL_1, remain valid.

Now let us re-do McC. There seems no better way to do it than through PoorMan, using ML_1 and retaining the definitions of t as p⊃p and NA as t⇒A. N isn't primitive any more, but defined; given that caution, we take as a basis for McC ⊃E as rule, and as axioms all formulas of ML_1 of the kinds A1-A4 and of the following kind:
A5. A⇒B.≡ .N(A⊃B)&(A≡B).
A5 at least has the merit for what it's worth, of locating the--shall we say, peculiarities--of ⇒ in a single axiom scheme that shows what's going on. To show that McC as reformulated is complete, use axioms A1-A4 to prove, essentially, the S5-translate of a given theorem C. (For reasons essentially as before, we may assume C a basic disjunction, which now means that each disjunct is of the form p, -p, A⇒B, or -(A⇒B).) The parenthetical remark makes replacement of equivalents to get C from its translate via A5 an essentially truth-functional matter, thus ending the proof of the completeness of McC when formulated with A1-A5, and ⊃E. (Since McCall 1975 has 14 axioms, the detour through PoorMan leaves us nine axioms to the good, only the last of which even looks unfamiliar.)

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THE THEORY OF LIMP IMPLICATION*

Zane Parks

It is commonly urged that the material conditional does not capture the sense of 'if---then' in ordinary language, and so far I agree, but the usual treatment of conditionals goes on to suggest that something more is meant by 'if---then' and here I must demur. Surely it is just the opposite --that something less is meant. The argument for this seems to me overpowering--namely, that people frequently (sincerely and, dare I add, truly) assert 'if---then' statments with true antecedents and false consequents. Surely the common man does not err so frequently as the proponents of the material conditional would have us believe.

What I propose to do is develop a theory of conditionals that does the common man justice. Thankfully, we can salvage something from preceding work on conditionals. It turns out that alethic modal logic provides just the right foundation for our theory: the theory of limp implication. Limp implication as defined below seems to be a fairly close approximation to 'if---then' as used in ordinary language. Of course, further refinement is possible. One might want to take irrelevance into account.

We shall say that $\underline{A}$ limply implies $\underline{B}$ just in case $\underline{A}$ possibly materially implies $\underline{B}$, or, in symbols:

$$\underline{A} \curvearrowright \underline{B} =_{df} \Diamond(\underline{A} \supset \underline{B}).$$

Although it would be desirable to investigate limp implication in other systems of modal logic, attention is restricted here to $\underset{\sim}{\underset{\sim}{S5}}$. (One particularly interesting alternative is $\underset{\sim}{\underset{\sim}{S6}}$ in which we would have, for example, $\Diamond(\underline{A} \curvearrowright \underline{B})$ as a theorem.) Without further ado, we list some strong and natural theorems

of the theory of limp implication.

(1) $\underline{A} \curvearrowright \underline{A}$ (Identity)

(2) $(\underline{A} \curvearrowright .\underline{A} \curvearrowright \underline{B}) \curvearrowright .\underline{A} \curvearrowright \underline{B}$ (Contraction)

(3) $\underline{A} \curvearrowright .\underline{A} \curvearrowright \underline{B} \curvearrowright \underline{B}$ (Assertion)

(4) $\underline{A} \curvearrowright (\underline{B} \curvearrowright \underline{C}) \curvearrowright .\underline{B} \curvearrowright .\underline{A} \curvearrowright \underline{C}$ (Permutation)

Next, some unnatural theorems:

(5) $\underline{A} \curvearrowright \underline{B} \curvearrowright \underline{A} \curvearrowright \underline{A}$ (Peirce)

(6) $(\underline{A} \& \sim \underline{A}) \curvearrowright \underline{B}$ (Melissus)[1]

(7) $\underline{A} \curvearrowright .\underline{A} \curvearrowright \underline{A}$ (Mingle)

Finally, a nontheorem:

(8) $\underline{A} \curvearrowright \underline{B} \curvearrowright .\underline{B} \curvearrowright \underline{C} \curvearrowright .\underline{A} \curvearrowright \underline{C}$ (Transitivity)

Indeed, addition of (8) to the pure theory of limp implication would presumably collapse (inflate?) limplication to material implication since (5), (7), and (8) together with modus ponens generate the pure theory of material implication (see Łukasiewicz [3]). It might be noted that modus ponens does not hold for limp implication except in the weakened form: from $\underline{A}$ and $\underline{A} \curvearrowright .\underline{B} \curvearrowright \underline{C}$, to infer $\underline{B} \curvearrowright \underline{C}$. Hereafter, this rule is referred to as $\curvearrowright$E.

Failure of (8) will perhaps explain the failure of Lewis' argument [2, p. 250], to convince some that

(9) if $\underline{A} \& \sim \underline{A}$, then $\underline{B}$

is a logical truth since that argument requires that implication be transitive, but implication (in the sense of limp implication) isn't.[2] So, the argument is invalid. Nevertheless, the conclusion is true in the sense that (9) construed as (6) is logically true.

It might be instructive to consider the relationship between material and limp implication. Obviously, the former implies the latter, that is,

(10) $\underline{A} \supset \underline{B} \rightsquigarrow . \underline{A} \rightsquigarrow \underline{B}$.

Perhaps less obvious though is the fact that the converse obtains also, that is,

(11) $\underline{A} \rightsquigarrow \underline{B} \rightsquigarrow . \underline{A} \supset \underline{B}$.

So that, in a sense, material and limp implication are equivalent. This, however, does not show limp implication to be superfluous since equivalence in the sense of mutual limp implication does not validate replacement.

We close by listing some open problems:

1. What is the relationship between the theory of limp implication and determinism? Note that

(12) $\underline{A} \rightsquigarrow \Box \underline{A}$

is a theorem where the underlying modal logic is at least as strong as von Wright's system $\underset{\sim}{M}$.

2. Since our primary concern is with implication, it would be of interest to isolate the pure theory of limp implication for various systems of modal logic.

3. Is γ, that is, the rule: from $\sim \underline{A}$ and $\underline{A} \vee \underline{B}$, to infer $\underline{B}$, admissible in the theory of limp implication as based on (say) $\underset{\sim\sim}{S5}$?

NOTES

* I am indebted to Mickey Perloff for helpful suggestions concerning notation and terminology. This paper was written in 1970 and has not been revised in any interesting way since then.

1. At Physica $185^{a}10$, Aristotle says "the argument of Melissus is gross and palpable and offers no difficulty at all; accept one ridiculous proposition and the rest follows--a simple enough proceeding" (Oxford translation).

2. Nontransitivity of implication (or entailment) has also been maintained by the English logician von Wright-Geach-Smiley. See the discussion

of the von Wright-Geach-Smiley criterion for entailment in [1] and references therein.

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ON A PURPORTED IMPLICATION

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In [1], Parks has proposed a modal analysis of implication, which he calls "limp implication", which, he argues, avoids alleged difficulties in the theory of the material conditional. Parks, alas, is not so explicit, but were he to follow Lewis in isolating "paradoxes" of material implication he would find that the chief of them is, on his motivating considerations,

1) p & (p-->q) ==> q,

where --> is limp (i. e., for Parks, ordinary) implication and ==> is material implication. For Parks, in deference to what he calls "ordinary" usage, wants to allow the possibility that both a limp implication and its antecedent are true without its consequent being thereby true.

Evidently, however, he has fallen into use-mention confusion. Implication, in the clearest sense, is a relation between sentences. The conditional, on the other hand, is a connective; i. e., an expression of the sort that, applied to sentences, yields a sentence. Moreover the conditional, as we all know, is truth-functional through and through. If then our common understanding of implication is not to be equated with the simple truth of a corresponding conditional, it must be the case that some other, structural factor is at work determining which implications hold.

Now it is certainly the case that for very many sentences 'p' and 'q', we want to assert both that 'p' is true and that 'p' implies 'q', without asserting also that 'q' is true. Indeed, logic would lead us into all sorts of difficulties if we did not have this desire. Let us, accordingly, go right ahead. But let us not be misled, or seduced, into proposing a variant conditional, which we would not at any rate understand and which would only confuse us about the number of planets.

I note in conclusion that where we are clear, namely with respect to implications between truth-functional expressions,

Parks' theory is just the ordinary one. I. e., where A and B are truth-functional,

2) A-->B is a theorem of Parks' system iff A==>B is a classical tautology,

which is enough in itself to show that he has not discovered anything new or interesting.[1]

NOTES

1. The author is indebted to W. V. Quine for many helpful conversations.

REFERENCES

[1] Parks, op. cit.

A New "Classical Relevant Logic"

The system C of Pottinger (1974) has as primitive logical constants => (entailment), $\dashrightarrow$ (S_4 strict implication), $\to$ (relevant implication), $\supset$ (intuitionist implication), & (intuitionist conjunction), $\vee$ (intuitionist disjunction) and Λ (absurdity). In § 8 of Pottinger (1974) it was pointed out that addition of either $\neg\neg A \to A$ or $\neg B \to \neg A \to. A \to B$ to C collapses $\to$ into material implication and that addition of either $\neg\neg A \Rightarrow A$ or $\neg B \Rightarrow \neg A \Rightarrow. A \Rightarrow B$ to C collapses => into S_4 strict implication. ($\neg A$ is defined to be $A \supset \Lambda$.) Pottinger (1974) left open the question whether adding excluded middle to C would also cause a collapse of relevance. It turns out that the answer to this question is "No!" It follows that the system obtained by adding excluded middle to C is a new "classical relevant logic". Proofs of these results will now be sketched. Because Pottinger (1974) has not yet been published, appropriate background information about C will be presented before proceeding with the proof sketches.

If i is a positive intiger, then ${}^{i}A$ is an indexed formula. M is to be either $\Box$ or the empty expression. H, H_1, ... are to be things of one of the forms ${}^{i}A^{M}$ or ${}^{i}A^{a}$, where a is a finite set of positive integers and $i \notin a$. P, Q, R, S, P_1, ... are to be lists of the form $H_1, \ldots, H_k$ ($k \geq 0$), where:

(i) for $i < j \leq k$, if ${}^{\ell_i}A_i$ is the indexed formula of H_i and ${}^{\ell_j}A_j$ is the indexed formula of H_j, then $\ell_i \neq \ell_j$

(ii) for $j \leq k$, if H_j has the form ${}^{\ell}A^{a}$ and $m \in a$, then for some $j_1 < j$, H_{j_1} has the form ${}^{m}B^{M}$.

Sequents have the form $P \vdash A_a$.

'X' will be used as a variable taking $\Rightarrow$, $\not\Rightarrow$, $\rightarrow$, and $\supset$ as values. $\Rightarrow$ and $\not\Rightarrow$ are strict and $\Rightarrow$ and $\rightarrow$ are relevant. Define:

(i) Λ is strict.

(ii) If X is strict, then AXB is strict.

(iii) If both A and B are strict, then A&B and A∨B are strict.

The system C_S is defined as follows:

A1: $P, {}^{i}A^{M} \vdash A_a$

where $a \subseteq \{i\}$

A2: $P, {}^{i}A^{b} \vdash A_a$

where $a \subseteq b$

Rules:

Reit: Inferences of the form

$$\frac{P \vdash A_a}{P, H \vdash A_a}$$

are permitted, provided that if H has the form ${}^{i}B^{\Box}$, then A is strict.

XE: Inferences of the from

$$\frac{P \vdash A_c \qquad P \vdash AXB_d}{P \vdash B_a}$$

are permitted, where:

(1) if X is relevant, then $a = c \cup d$

(2) if X is not relevant, then $a = d$.

&E: $\dfrac{P \vdash A \,\&\, B_a}{P \vdash A_a}$ $\qquad$ $\dfrac{P \vdash A \,\&\, B_a}{P \vdash B_a}$

$\vee$E: Inferences of the form

$$\frac{P, {}^{i}A^{a},\ Q \vdash C_{a_1} \quad P, {}^{i}B^{a}, Q \vdash C_{a_2} \quad P \vdash A \vee B_{a_3}}{P, Q \vdash C_{a_1 \cap a_2}}$$

are permitted, provided $a \subseteq a_3$.

$\wedge$E: $\dfrac{P \vdash \wedge_a}{P \vdash A_a}$

XI: Inferences of the form

$$\frac{P, {}^{i}A^{M} \vdash B_a}{P \vdash A \times B_{a - \{i\}}}$$

are permitted, provided that:

(1) if X is relevant, then $\{i\} \subseteq a$

(2) if X is strict, then M is $\Box$.

&I: $\dfrac{P \vdash A_a \quad P \vdash B_b}{P \vdash A \,\&\, B_{a \cap b}}$

$\vee$I: $\dfrac{P \vdash A_a}{P \vdash A \vee B_a}$ $\qquad$ $\dfrac{P \vdash B_a}{P \vdash A \vee B_a}$

$\wedge$I: $\dfrac{P \vdash \wedge_a \quad P \vdash A_b}{P \vdash \wedge_{a \cup b}}$

In Pottinger (1974) it was shown that if => is the only logical constant occurring in A, then $\vdash$ A is a theorem of C_S iff A is a theorem of E and that if $\rightarrow$ is the only logical constant occurring in A, then $\vdash$ A is a theorem of C_S iff A is a theorem of R. These results were obtained as corollaries of the Normalization, Subformula, and Separation theorems, which will now be stated in preparation for the proof sketches to be given below.

The major premiss of an XE, & E, $\vee$E, or $\wedge$I is the rightmost premiss of that inference. Let *s* be an occurrence of the sequent s in the derivation $\underset{\sim}{D}$. *s* is maximal iff *s* is derived from an XI, &I, $\vee$I, or $\wedge$ E via a series of Reits and $\vee$E's and *s* is the major premiss of an XE, &E, $\vee$E, or $\wedge$I in $\underset{\sim}{D}$. $\underset{\sim}{D}$ is normal iff no sequent occurrence in $\underset{\sim}{D}$ is maximal.

Theorem 1 (Normalization Theorem): Given any derivation $\underset{\sim}{D}$, one can find a normal derivation $\underset{\sim}{D}'$ s.t. $\underset{\sim}{D}$ and $\underset{\sim}{D}'$ have the same endsquent.

Theorem 2 (Subformula Theorem): If $\underset{\sim}{D}$ is a normal derivation, s is a sequent occurring in $\underset{\sim}{D}$, and A is a formula of s, then A is a subformula of some formula of the endsequent of $\underset{\sim}{D}$.

Corollary 3 (Separation Theorem): If $\vdash$ A is a theorem of C_S, then there is a normal C_S derivation $\underset{\sim}{D}$ s.t. $\vdash$ A is the endsquent of $\underset{\sim}{D}$ and:

(a) if the logical constant K does not occur in A, then no inference of $\underset{\sim}{D}$ is either a KE or a KI.

(b) if no logical operator occurring in A is strict, then no sequent occurring in $\underset{\sim}{D}$ has the form P, ${}^{i}_{B}\Box$, Q $\vdash C_a$.

(c) if $\vee$ does not occur in A, then no sequent occurring in $\underset{\sim}{D}$ has the form P, ${}^{i}_{B}a$, Q $\vdash C_b$.

Now let CC_S be the system obtained from C_S by adding all sequents of the form $\vdash$ A $\vee \neg$ A as axioms.

Theorem 4: Neither $\vdash p \Rightarrow . q \Rightarrow q$ nor $\vdash p \rightarrow . q \rightarrow q$ is a theorem of CC_S.

Sketch of proof: It is easy to prove $\vdash p \Rightarrow (q \Rightarrow q) \Rightarrow . p \to . q \to q$ in C_S, so it suffices to show that $\vdash p \rightarrow . q \rightarrow q$ is not a theorem of CC_S.

Suppose for reductio that $\vdash p \to . q \to q$ is a theorem of CC_S. Then there exist $A_1, \ldots, A_n$ and $i_1 \ldots, i_n$ s.t. $i_1 A_1 \vee \neg A_1, \ldots, i_n A_n \vee \neg A_n \vdash p \rightarrow . q \rightarrow q$ is a theorem of C_S. Also, all sequents of the following forms are theorems of C_S: (1) $P \vdash A \supset B \supset . \neg B \supset \neg A$, (2) $P \vdash \neg\neg A \supset A$, and (3) $P \vdash \neg\neg (A \vee \neg A)$.

It follows that $\vdash \neg\neg (p \rightarrow . q \rightarrow q)$ is a theorem of C_S.

Let C_{S_1} be the system which is obtained from C_S by requiring that the A in $\wedge$ I and $\wedge$ E be atomic. It can be shown via an induction on the complexity of A that the unrestricted forms of $\wedge$ I and $\wedge$ E given above are derivable in C_{S_1}, and inspection of the appropriate proofs of Pottinger (1974) leads to the conclusion that the Normalization, Subformula, and Separation theorems hold for C_{S_1}. A straightforward argument based on these theorems shows that $\vdash \neg\neg (p \to . q \to q)$ is not a Theorem of C_{S_1}, and Theorem 4 follows from this.

It is likely that the proof of Theorem 4 can be generalized so as to show that the $\Rightarrow, \neg, \rightarrow$ - fragment of CC_S is not larger than the $\Rightarrow, \neg, \rightarrow$ - fragment of C_S. (I am currently trying to do this.)
A more difficult question, which cannot be answered by generalizing the proof of Theorem 4, is "What happens if all sequents of the form $P \vdash A \vee \neg A$ are added to C_S as axioms?" (This would be equivalent to adding all sequents of the form $\vdash \Box (A \vee \neg A)$ as axioms, where $\Box B$ is defined to be $B \Rightarrow B \Rightarrow B$).

It follows Theorem 4 and the remarks of the first paragraph that none of the following sequents a theorem of CC_S:

(1) $\vdash \neg\neg p \rightarrow p$

(2) $\vdash \neg q \rightarrow \neg p \rightarrow . p \rightarrow q$

(3) $\vdash p \rightarrow q \vee \neg q$

(4) $\vdash \neg p \supset q \leftrightarrow p \vee q$

(5) $\vdash \neg (\neg p \,\&\, \neg q) \leftrightarrow p \vee q$

(6) $\vdash \neg (\neg p \vee \neg q) \leftrightarrow p \,\&\, q.$

On the other hand, everything of the following forms can be proved in CC_S:

(7) $\vdash A \supset B \leftrightarrow \neg A \vee B$

(8) $\vdash \neg (A \& B) \leftrightarrow \neg A \vee \neg B.$

Since everything of the form $\vdash A \,\&\, \neg A \rightarrow B$ can be proved in C_S and since CC_S evidently contains all classical tautologies in $\supset$, &, $\vee$, and $\wedge$, the non-theoremhood of any of (1), (3), (4), or (5), above, suffices to show that CC_S is a "classical relevant logic" the $\rightarrow$, &, $\vee$, $\neg$ - fragment of which is different from the system CR of Myer and Routley (1973). It is easy to see that if all instances of any of (1), (3), (4), or (5) were added to CC_S, then everything of the form $\vdash \neg\neg A \rightarrow A$ would be a theorem of the extended system and $\rightarrow$ would collapse into material implication. Hence, the divergence between CC_S and CR is necessitated by properties of C_S.

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11/5/76

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ULTRALOGIC AS UNIVERSAL?

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The central idea (see footnote 13) is that some sets are inconsistent because, when comprehension is applied, they turn out to have inconsistent properties. Thus, where x is a multiplicity, x is directly inconsistent iff $(Px)(x \in z \ \& \ x \notin z)$. But this definition does not, on its own, ensure that x is a multiplicity. By making use of extensionality, a superior definition, which implies multiplicity in the sense of having members, results:

$$x \text{ is d-inconsistent} =_{Df} (Pz)(z \in x \ \& \ z \notin x).$$

Then the Russell set, R, is d-inconsistent, since $R \in R$ and $R \notin R$. Similarly presumably, by paradox arguments, the sets consisting of all cardinals and of all ordinals are both d-inconsistent.

But direct inconsistency is insufficiently comprehensive. For if a set has an inconsistent part or an inconsistent element then presumably it is inconsistent in virtue of that. Cantor wants, in fact, to go further and to count a set as inconsistent if it can even be put into 1-1 correspondence with an inconsistent set. What needs to be guaranteed by appropriate definitions, then, are the postulates:

$$x \text{ is d-inconsistent} \rightarrow I(x), \text{ i.e. } x \text{ is inconsistent;}$$

$$I(x) \ \& \ (x \in y \vee x \subset y) \rightarrow I(y) \ .$$

(only the membership case causes a definitional problem, requiring introduction of the $\in$-ancestral.) It follows that the set U of all sets is inconsistent, since it has R as a part, and, what is more disputable, that the sets {R}, {{R}}, ..., are all inconsistent.

It is evident however that the important but laborious task of seeing what can, and, equally important, what cannot, be accomplished in DST lies, for the most part, ahead. What has been presented is but a small beginning in that enterprise.

§9. Ultramodal mathematics: arithmetic. In order to sustain the ultramodal challenge to classical logic it will have to be shown that even though leading features of classical logic and theories have been rejected, one can still get by. In particular, it will have to be shown that by going ultramodal one does not lose great chunks of the modern mathematical megalopolis. The strong ultramodal claim - not so far vindicated - is the expectedly brash one: we can do everything you can do, only better, and we can do more. More, because there are whole mathematical cities that have been closed off and partially abandoned because of the outbreak of isolated contradictions, notably theories of the very small, infinitesimals, and theories of the very large, Cantor's set theory. Admittedly there have been modern restorations of apparently consistent suburbs of these theories, but the life of these cities has vanished and they have become like modern restorations of ancient cities, like modern Balbus or Leptis Magna, mostly just patched-up ruins visited by tourists.

The question of the recovery of modern mathematics, or for that matter of classically-derivable mathematics, is not so far a very precise one, because there are different notions of recovery. In one

sense, that of a liberal postulate theory, where any requisite postulates for a topic may be pressed into service, the whole of exactly-formulated modern mathematics is rather trivially obtainable simply by throwing on the requisite postulates. Since classical sentential logic is obtainable in this way, just by adding the paradox postulate $A \rightarrow . B \rightarrow A$, it is immediate that any theory that can be represented by classical postulate theory (as explained, e.g. in Church [23]) can be obtained in ultramodal postulate theory as well. But the procedure, of adding on sufficiently many further postulates, may lead to infringements of other prized properties, such as consistency, relevance, and so on. It may be that such a postulational recovery is the best that can be achieved for some mathematical theories, e.g., those that are shot through with irrelevance. But at least in the case of central mathematical theories, such as arithmetic and number theory, the ultramodal objective is to establish stronger types of recovery.

There are two sorts of relevant recovery of especial interest in the case of arithmetic, namely recovery within ultramodal set theory, and recovery, as far as possible, in a first order framework. The first sort is deeper, since it provides an analysis of arithmetic notions: but it is also a larger and more vexed enterprise - because many of the moves in classical set-theoretical reductions of arithmetic fail relevantly, because it presupposes much more, as the non-triviality of arithmetic comes to depend on the non-triviality of the underlying set theory, and because all the controversial and questionable features of ultramodal set theory get imported into arithmetic. But such a set-theoretic analysis is nonetheless important, since it should furnish a guide to the correctness of the principles assumed in the second sort of recovery, namely in first order relevant arithmetic. Once again, however, there is a stronger type of recovery to consider, that is to say a type of enthymematic recovery for classical first order Peano arithmetic formulated with material implication, $\supset$, as the entailment relation.

To establish such classical recovery in the case of arithmetic (which provides an important working example) there are two problems of independent interest. The first is the question of a properly entailmental axiomatisation of arithmetic, since the usual Peano postulates are unsatisfactory. The second matter, the main issue in practically every case involving enthymematic recovery, is whether the rule γ is derivable for ultramodal arithmetic. Proof of γ requires, in effect, a soundness and completeness result, and so includes a consistency proof. In particular, a proof of γ for ultramodal arithmetic would not merely establish the consistency of classical arithmetic, should it be consistent; it would also help circumscribe Gödel's second theorem.

Consider the problem of the proper axiomatisation of arithmetic first. Peano arithmetic reformulated with entailmental relations occupying the main implicational positions leads at once to results of dubious entailmental validity, such as:

$3 = 3 \rightarrow 19 = 19$,

$19 = 19 \rightarrow 1 = 1$,

$3 = 5 \rightarrow 9 = 9$, and, more generally,

(a) $m=m \leftrightarrow n=n$, and

(b) $t=n \rightarrow m=m$.

None of these are correct entailmental principles, I want to claim, yet they are almost immediate given correct identity principles (symmetry and transitivity) from the Peano postulates:

$P\alpha$. $n=n \rightarrow .\ n+1 = n+1$

(or $Nx \rightarrow .\ x=x \rightarrow x' = x'$, if the domain is not restricted to natural numbers) and

$P\beta$. $n+1 = n+1 \rightarrow .\ n=n$.

Proposition (a) follows from these two postulates by transitivity of identity using the fact that any natural number m may be represented as the m^{th} successor of zero, and then (b) follows from (a) since $t=n \rightarrow n=t$, whence $t=n \rightarrow (n=t)\ \&\ (t=n)$, and $t=n \rightarrow .\ n=n$. But propositions (a) and (b) are subject-restricted paradoxes. For according to (a) any correct numerical equation entails any other while (b) guarantees that any logical false equation with a numerical term entails any true numerical equation. And in Meyer's arithmetic (system R# of [15]) these results spread to more extensive paradoxes, such as that any correct numerical equation entails any theorem of arithmetic.[14]

Attempts to prove $P\alpha$ and the more general form

$P\alpha\gamma$ $n=n \rightarrow .\ n+m = n+m$

to which it leads within the framework of a relevant set theory suggest the requisite modifications to these principles. Just as the principle of Factor:

$p \leftrightarrow q \rightarrow .\ p\ \&\ r \leftrightarrow q\ \&\ r$

is corrected within the relevant framework here adopted through the principle

$p \leftrightarrow q\ \&\ r \leftrightarrow r \rightarrow .\ p\ \&\ r \leftrightarrow q\ \&\ r$,

so $P\alpha\gamma$ is corrected by reinstating a suppressed, because obviously correct, premiss $m=m$, giving

$n=n\ \&\ m=m \rightarrow .\ n+m = n+m$.

Similarly $P\alpha$ is corrected to

$P\alpha'$. $n=n\ \&\ 1=1 \rightarrow .\ n+1 = n+1$.

---

[14] There are other significant anomalies in R#. For example, there are puzzling discrepancies between the strength of addition and subtraction principles on the one hand and those of multiplication and division on the other. Thus while subtraction principles, such as $m+3 = n+3 \rightarrow m=n$, hold in entailmental form in R#, corresponding cancellation principles, such as $mX3 = nX3 \rightarrow m=n$ do not.

The Peano principle Pβ may be similarly corrected if first recast as $n=n \rightarrow n\dot{-}1 = n\dot{-}1$, a correct form being

Pβ'. $n=n \;\&\; 1=1 \rightarrow .\; n\dot{-}1 = n\dot{-}1$

The formal arithmetic DKA that emerges after these corrections adds to quantificational logic DKQ, formulated with 0, +, X, ' (i.e. zero, addition, multiplication and succession) as non-logical constants (and minus specifically contradictory axioms), the following non-logical axioms:

A1. $x=y \;\&\; \tau \rightarrow .\; x' = y'$

A2. $x' = y' \;\&\; \tau \rightarrow .\; x=y$

A3. $x=y \;\&\; y=z \rightarrow .\; y=z$

A4. $x=y \rightarrow y=x$

A5. $x' \neq 0$

A6. $x+0 = x$

A7. $x+y' = (x+y)'$

A8. $x \times 0 = 0$

A9. $x \times y' = (x \times y) + x$

and the following induction rule:

RMI $A(0),\ A(x) \rightarrow A(x') \multimap A(x)$,

where $\tau =_{Df} 1=1$, $1 =_{Df} 0'$, and $t\neq u =_{Df} \sim(t=u)$.

These arithmetic axioms may of course be equally well added to quantified relevant logics other than DKA, e.g. to system RQ, yielding an arithmetic RA. RA differs from Meyer's arithmetic R# (of [15]), which simply adopts arrow reformulations of the usual first order Peano axioms, in just the following respects:-

(i) the R# axiom $x=y \leftrightarrow x' = y'$ is replaced by A1 and A2;

(ii) the R# axiom $x=y \rightarrow .\; x=z \rightarrow y=z$ is replaced by A3 and A4.

If DKA is formulated with constant t, where t satisfies, as usual, the rule $A \leftrightarrow t \rightarrow A$, then since $\vdash t \rightarrow \tau$, A1 and A2 yield the enthymematic forms $x=y \supset x' = y'$ and $x' = y' \supset x=y$, where $\supset$ is the intuitionist-like implication defined by $A \supset B =_{Df} A \;\&\; t \rightarrow B$.

Whereas R# has been extensively investigated, there is much work to be done on DKA. Several mappings connecting R# and classical arithmetic are known, with the result that it is easily established, relying on classical results, that many classical theorems hold relevantly, that all recursive relations are expressible in R#, and accordingly, given the classical mythology (on which see Meyer [15]), that Gödel's first theorem holds for R#. Whether these results hold for DKA is not known; in particular, it is not known whether or not every recursive relation is expressible in DKA. Research thus far has been concentrated on establishing the admissibility of γ for DKA,

as this would settle in one blow a great many such open questions; but definitive results are still lacking.

§10. Another question of adequacy: consistency arguments. Though DST deliberately rejects an orthodox criterion of adequacy for a theory, namely negation consistency, it does not escape all consistency-style checks on adequacy. In particular, it would hardly be a satisfactory theory if it were trivial, if every assertion were a thesis. Similarly ultramodal arithmetic would be a worthless theory were it trivial, and even if non-trivial it would be decidedly unsatisfactory if it enabled the proof of arithmetically incorrect assertions such as $0=1$ or $0\neq 0$.

The consistency of ultramodal arithmetic is not in doubt, and not simply because it is a sub-system of classical arithmetic. There is an elementary non-triviality proof for ultramodal arithmetic which at the same time shows that the incorrect equations $0=1$ and $0\neq 0$ are not theorems.[15] The position with respect to DST is less clearcut, but there are, however, partial results, which show essentially that DST is non-trivial provided that the implication symbol, $\rightarrow$, does not occur in the right-hand side of the general comprehension axiom. Obviously this is a severe restriction on the fecundity of the theory, and research aimed at removing the restriction is proceeding. Meanwhile it is worth reporting partial results.

In both consistency arguments - that for arithmetic and that for dialectical set theory - the matrices of 3-valued system RM3, interpreted dialectically, will be appealed to (RM3 is discussed in ABE). Where $\sim$, & and $\rightarrow$ are primitive (with $\vee$ is defined as usual: $A \vee B =_{Df} \sim(\sim A \,\&\, \sim B)$) the requisite matrices of RM3 are these:-

&	t	n	f
t	t	n	f
n	n	n	f
f	f	f	f

$\sim$
f
n
t

$\rightarrow$	t	n	f
t	t	f	f
n	t	n	f
f	t	t	t

Where the matrix values are given arithmetic significance, namely t, n and f are identified respectively with +1, 0 and -1, the value of A & B is the minimum of the values of A and B (similarly $\vee$ takes the maximum of component values), the value of $\sim$A is the inverse of the value of A, i.e. $v(\sim A) = -v(A)$, and the value of $A \rightarrow B$ is the maximum of $-v(A)$ and $v(B)$ when $v(A) \leqslant v(B)$ and the minimum of $-v(A)$ and $v(B)$ when $v(A) > v(B)$.

The matrix for $\leftrightarrow$, defined $A \leftrightarrow B =_{Df} (A \rightarrow B) \,\&\, (B \rightarrow A)$, is also needed: it has value f for non-diagonal elements and value t for diagonal elements except when both A and B have value n in which case $v(A \leftrightarrow B) = n$.

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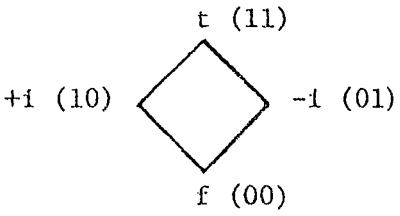
15 This finitary proof is a simple adaptation of an important elementary proof found by Meyer; as Meyer argues in [15] the proof is enough to undermine all the philosophical applications that have been made of Gödel's second theorem, and to imperil the scope of the theorem as well.

The matrices for &, ∨ and ~ are precisely those for Łukasiewicz's system $Ł_3$. RM3 differs from $Ł_3$ however in taking both t and n as designated values (that is, it is a C-system in the sense of SC). The explanation as to why n is designated lies in the intended dialectical interpretation of n as both true and false. Thus if A has value n then A is true: it may also be false, but since true it should be designated in the same say as when A has value t.

The quantifiers U and P of the quantificational extension RM3Q of RM3 behave like infinite conjunctions and disjunctions respectively, with $(Ux)A(x)[(Px)A(x)]$ taking the minimum [maximum] value of the arithmetic values assigned to A(x). More precisely, for a given assignment to the free variables other than x of A(x), $v((Ux)A(x))= t$ iff $v(A(x)) = t$ for every assignment of values to x; $v((Ux)A(x)) = f$ iff $v(A(x)) = f$ for some assignment of values of x; and $v((Ux)A(x)) = n$ otherwise.

There is an appealing intuitive case for favouring 4-valued matrices, which allow for incompleteness as well as inconsistency, rather than 3-valued matrices for the consistency arguments. Then n - which so far can either be interpreted as a neuter value, or as an underdetermined value, -i, or as an overdetermined value, +i - would split into the two values, +i and -i, with +i construed as both true and false, i.e. {t, f} and -i construed as neither true nor false, i.e.∧. The matrices for connectives &, ∨, ~ may be computed piecewise from the Hasse diagram:



where the 4-element lattice is represented, as shown, as a product of two 2-element Boolean algebras. It follows that conjunction and disjunction are again given as minimum and maximum values, and negation has the matrix:

~	t	+i	-i	f
	f	+i	-i	t

These matrices play an important part in the theory of entailment (see e.g. ABE, p. 161 ff.) and the whole semantical analysis of entailment can be built upon them. But while the arguments which follow can be carried through in terms of these 4-valued matrices, the fourth indeterminary value is not so far needed; simpler 3-valued arguments suffice.

The integers modulo 2 provide the domain for the model in Meyer's consistency proof for relevant arithmetic. For the proof which follows - which unlike Meyer's proof refutes $0 \neq 0$ - the domain is given by the sequence:-

(S) 0, 1, 2, 1, 2, 1, 2, ... ,

where 1=0' and 2=0".[16] In short, the elements are those of {0, 1, 2}, with 1 the successor of 0 and 2, and 2 the successor of 1. Matrix-values are assigned to the atomic wff, numerical equations, as follows: v(0=0) = t, each of 1=1, 2=2, 0=2, and 2=0 is assigned value n, while 1=2, 2=1, 0=1, 1=0 are assigned value f, i.e. incorrect numerical equations have value f while correct equations - according to (S) - are assigned either value t or n. The values of complex wff are then determined through the RM3Q modelling given, with operations determined faithfully, i.e. v(x+y) = v(x)+v(y), v(x X y) = v(x) X v(y) and v(x') = (v(x))' for operations in sequence (S). It follows at once that v(0≠0) = v(0=1) = f. The modelling likewise refutes every other incorrect numerical equation. But every theorem of DKA and of RA is assigned a designated value, that is, where B is a theorem of DKA, v(B) = t or v(B) = n, by induction on the proof of B. For it is a matter of direct verification that each axiom takes a designated value and that the rules preserve designated values.

Theorem. Neither 0≠0 or 1=0 is a theorem of DKA (or of RA); hence DKA is absolutely consistent.

Since the proof of consistency is a finitary one it can presumably be represented in DKA itself. Thus DKA, like R#, escapes Gödel's second theorem. Furthermore, these results for relevant arithmetics, if they do not refute outright, certainly cast serious doubt on, Gödel's sweeping claim ([24], p. 616)[17] that

> ... it can be proved rigorously that in every consistent formal system that contains a certain amount of finitary number theory there exist undecidable arithmetic propositions and that, moreover, the consistency of any such system cannot be proved in the system.

For by 'consistency' Gödel means absolute consistency (see [24], p. 614, footnote 63), and by 'contains a certain amount of finitary number theory' he appears to mean that every recursive relation be expressible

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16 In both classical arithmetic and Meyer's R# if one could prove both A and ~A for some A one would be able to prove 0≠0, i.e. negation consistency is tantamount to the refutability of 0≠0. In each case this is due to paradoxical features of the arithmetic. In DKA however negation consistency is not deductively equivalent to the refutability of any incorrect numerical equation or inequation.

17 This is by no means the only respect in which Gödel's claim is too sweeping. There are no rigorous proofs without assumptions, and Gödelian-style proofs of incompleteness and undecidability make rather large, and unstated, assumptions about what is admissible in the metalanguage in terms of which the proofs of incompleteness are carried out. Without these assumptions the proofs fail. In particular, if all content self-dependence is outlawed in genuine statements - including of course that induced through translations such as Gödel numbering - as would be done by a thorough-going uniform resolution of paradoxes, then the standard assumptions do fail; see [25].

(cf. [24], p. 617); yet R# meets Gödel's conditions, and permits demonstrations of its own consistency. In fact a reading of the fine print of Gödel's 'Postscriptum' ([34], p. 73) reveals that Gödel's only warrant for his sweeping claim is the much more limited result established by Hilbert and Bernays for classical arithmetic. Gödel's assurance, that 'the proof carries over almost literally to any system containing, among its axioms and rules of inference, the axioms and rules of inference of number theory', appears to be quite unwarranted.

A fuller understanding of the scope and status of Gödel's consistency theorem awaits, however, the outcome of investigations as to the admissibility of rule γ of material detachment in relevant arithmetics. For if γ is admissible in, i.e. is a derived rule of, some appropriate relevant arithmetic UA, then UA will include classical first order arithmetic on direct translation, and will in fact be a non-trivial recurisvely axiomatisable extension of classical arithmetic. On all standard accounts[18] the non-triviality of UA cannot be proved from the postulates of UA; but it can be. A proof of γ would also be important in other ways; it would provide an adequacy result of a sort for UA, through showing that UA contained at least as much number theory as classical first-order arithmetic; and moreover it would furnish a new consistency proof for classical arithmetic.

Another important open question, along with the admissibility of γ for relevant arithmetic, is the issue of the non-triviality of DST. The best results that have been obtained so far in this direction can be summed up as follows:-

Theorem. Where LQ is any quantificational logic included in RM3Q, then the set theory obtained by adding to LQ the general comprehension axiom, GCA, subject to the proviso that the implication symbol arrow does not occur on the right-hand side of GCA, and the extensionality rule, Ext. R, is non-trivial.

This theorem, obtained by elaborating the persistence results of Brady [12] and [28], falls short of the desired non-triviality result for DST in two respects: less seriously in the weakened form of extensionality, and more seriously in the exclusion of formulae built up using $\rightarrow$ from set determination in GCA. The latter restriction, effectively to extensionally determined sets, is undesirable: there is no reason why sets should not be intensionally specified, and indeed they commonly are essentially so specified. There is some promise that, by combining use of the 3-valued matrices with world semantics

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[18] For example, Shoenfield [26], p. 213:

> The general conclusion is that if an axiom system contains as much number theory as P then one cannot prove the consistency of that axiom system from the axioms of that system.

It might be thought that Shoenfield's argument accordingly showed that γ is inadmissible for any UA. But it does not because the argument given turns on classical features, such as the conflation of absolute and negation consistency, and, more important, demands the admissibility of γ for every recursive extension of UA, a most unlikely assumption.

for DKQ, that is by treating each world in a 3-valued fashion and applying persistence methods with respect to each world, the restriction on CCA in the non-triviality theorem can be lifted. Contrary to popular assumption, the impossibility of such a non-triviality result has not been demonstrated by Gödel's second theorem: as in the case of relevant arithmetic, so with DST, we are beyond the bounds of the validity of Gödel's famous claim - indeed dialectical set theories are not seriously hampered by Gödel's first theorem either.

That is where the ultramodal program in the foundations of mathematics stands, and that is where we shall leave that part of the program and turn to explications of more philosophical concern.

§11. Content and semantic information. The qualitative theory of semantic information is based on the analysis of content. But the classical theory of logical content, as worked out in most detail by Carnap [31], is shot through with paradoxical results, e.g. all necessary truths have the same content, namely none, and likewise all logical falsehoods have the same logical content. These paradoxes are of course derivative from the account of entailment, explicated as L-implication, which Carnap presupposes; that is, the paradoxes are derivative from the underlying paradoxes of strict implication, which in turn depend on the limitation to state descriptions or possible states-of-affairs. By changing the semantical base, but preserving essentially the classical definitions of content Carnap uses, and their intuitive bases, all these paradoxes can be avoided. (The account which follows is an enlargement of that in [14] to deal with higher degree formulae and also with quantified formula, i.e. the restriction to truth-functional formulae is removed. The account in [58] can be similarly enlarged.)

A condition of adequacy on any account of logical content, or information, is that it leads to the results that A entails B iff (the meaning of) B is included in the meaning of A, and (the meaning of) B is included in the meaning of A iff the content of A includes the content of B.

Part of the argument for this condition is simply this:- Whenever A entails B, A asserts all that is asserted or meant by B and perhaps more. Conversely, when A asserts all that is asserted by B, the meaning of B is included in the meaning of A, so A entails B. Now symbolising 'the content of A' by 'c(A)', part of the condition is secured by defining: (the meaning of) B is included in the meaning of A as $c(B) \subseteq c(A)$ - provided only that 'c(A)' can be defined in turn so that contents can be appropriately ordered by an inclusion relation. But, by the semantics for entailment (in a typical model), $A \rightarrow B$ iff for every (deductive) situation a if A holds in a then (materially) B holds in a, i.e., canonically, iff for every a if B is not in a then A is not in a; i.e. iff $\{a:B \text{ is not in } a\} \subseteq \{a:A \text{ is not in } a\}$.

Comparison of this equivalence with

$$A \rightarrow B \text{ iff } c(B) \subseteq c(A)$$

reveals that all the desired features follow if we connect

c(A) with {a:A does not hold in a} ,

that is, the content of A with the class[19] of worlds where A does not hold. The intuitive point of this connection - essentially that proposed by Popper [32] and exactly that adopted by Carnap [31] - is brought out by inserting these auxiliary definitional stages:

set-up a refutes $\Delta =_{Df}$ Δ does not hold in a;

$c(\Delta) =_{Df} \{a:a \text{ refutes } \Delta\}$,

i.e. the logical content of Δ is given by the situations which refute Δ or which Δ excludes. Incidentally the definition has been widened, by use of metavariable Δ, to take account of the content of all formulae including ill-formed formulae. The definition of refutation leads at once to such classical results as that if A entails B and a refutes B then a refutes A. As usual we say that A has no logical content iff $c(A) = \Lambda$, i.e. iff no world rules it out, and otherwise that A has some logical content; and that A has total logical content if $c(A) = V$, i.e. the class of all worlds (in the canonical model). Then, of course, an assertion has no content if no world refutes it, and total content if every world refutes it. Thus ill-formed formulae have no logical content. Since the class of worlds which refute A & B includes the class which refute A, $c(A) \subseteq c(A \,\&\, B)$. It is this result in particular, as Popper noticed, which warrants regarding c as a content notion: for there is a good sense in which A & B generally says more than A. But if A demonstrably entails B then B does not add anything to A, i.e. $c(A \,\&\, B) \subseteq c(A)$. Indeed the fact that $A \rightarrow B$ is provable iff $c(A \,\&\, B) = c(A)$ - a principle commonly adopted, e.g. by Halldén, to define entailment - provides yet a further warrant for regarding c as the logical content notion.

Moreover several desirable principles follow, in particular:-

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[19] Though the familiar class terminology has been adopted to make comparisons easy, a treatment of content as a property of statements would be more apposite. In fact everything done is compatible with such a treatment, which can be simply obtained by reconstruing {H:A is not in H} as λH(A is not in H) where λ provides property abstraction, or as Θ{H: A is not in H} where Θ is a suitable 1-1 function.

The exclusion account of content given is by no means the only notion of content that has figured in philosophical theories. Another important notion is that of signification: s(A) is defined as the class of assertions entailed by A. Semantically,

$s(A) = \{C:c(C) \subseteq c(A)\}$

Since

$c(A) = \cup\{c(C):c(C) \subseteq c(A)\}$,

it is evident that these explications are closely related attempts to capture the same determinable: content. My feeling is that the more syntactically oriented signification notion is the less satisfactory attempt.

Every wff has some content; thus every tautology has some content.

No wff has total content; so no contradiction has total content.

If A and B are disjoint wff, then they have distinct (non-inclusive) contents.

In particular, then, any two tautologies with distinct variables have distinct contents, and similarly distinct contradictions.

By contrast, the same definition of logical content based on a strict implicational relation leads to such counter-intuitive results such as that some assertions, namely necessarily true ones, have no content, whereas others, the negations of necessary assertions, have total content (see, e.g., Carnap [31]). Thus, strict implication, and likewise its metalinguistic formulations, such as L-implication, far from capturing a natural and inevitable account of logical content and necessary truth, embodies positivistic views as to the nature of necessary truth. This view of necessary truths as without content can also be explained as following from the view that they can always be suppressed. For using Halldén's principle, an assertion which does not add anything to any assertion either has no content or its content is included in that of every assertion. But some assertions are quite disjoint in meaning from others. Therefore necessary truths, like ill-formed formulae, have no logical content. Conversely, of course, the view that necessary truths have no content has been used to prop up the claim that they can be suppressed.

From a communicational viewpoint the strict account is ridiculous. A person transmitting necessary truths over some channel, e.g. through logic textbooks, is not sending no information and the less the higher the ratio of theorems; nor is the receiver getting no information. Likewise it is more than merely "strange" that a self-contradictory assertion - one which no "ideal" receiver would accept - is regarded as carrying with it the most inclusive information. Despite the Carnap-Bar-Hillel strict theory of information, contradictions are commonly not so vastly informative.

Inclusion of meaning - in the sense of content - provides an account of meaning connexion which answers Bennett's demand ([47], p. 214) that where there is an entailment there is a meaning connexion but that for some B there is no meaning connexion between A & ~A and B. Indeed these desirable principles follow from the account of inclusion of meaning given:

No wff is part of the meaning of every wff; in particular no tautology is part of the meaning of every wff.

No wff includes in its meaning every proposition; in particular no contradiction includes in its meaning every proposition.

For sentential wff A and B, B is included in the meaning of A only if B shares a sentential letter with A.

Where propositional identity is defined as coinclusion of logical content, i.e. as identity of logical content - a not implausible, but still less than adequate, proposal - it follows further that tautologies are propositionally distinct from other tautologies when they contain distinct propositional letters. Similarly for contradictions. But using similar definitions in terms of strict implication one would obtain the disastrous result that there is just one necessary proposition, which is included in the meaning of every proposition, and just one contradiction, which includes in its meaning every proposition.

In communication it is important to say not only what information a message or experiment has supplied but also how much. Hence in addition to an explication of content, or information, an explication of amount of information is sought.

The theory of semantic information is that part of the full theory of information that is concerned with the information and amount of information carried by an assertion, not that concerned with the information a sender of a message intended to convey by transmitting the message nor with the information a receiver actually obtained from the message - these issues are said to belong to pragmatics. The main parameters of the full theory of communication are nicely exhibited in Lasswell's slogan: who sends what in which channel to whom with what effect. The semantic theory is concerned only with what and what amount of information.

A major defect of the standard theory of semantic information, based on the work of Carnap and Bar-Hillel, is that under all the explications of information considered

> the amount of information carried by the sentence '17 x 19 = 323' is zero and ... the amount of information of 'The three medians of the sides of a plane triangle intersect in one point', relative to some set of sentences serving as a complete set of axioms for Euclidean geometry, is likewise zero ([33], p. 223).

Carnap and Bar-Hillel attempt to minimize[20] this problem by introduction of an "ideal receiver":

> The semantic information carried by a sentence with respect to a certain class of sentences may well be regarded as the 'ideal' pragmatic information which the sentence would carry for an 'ideal' receiver whose only empirical knowledge is formulated in exactly this class of sentences. By an 'ideal' receiver we understand, for the purposes of this illustration, a receiver with a perfect memory who 'knows' all of logic and mathematics, and together with any class of empirical sentences, all of their logical consequences.

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20 An even shabbier ploy has been invoked recently by Hintikka who requires, in order to save his logic of knowledge, 'epistemically perfect worlds' 'inhabited solely by deductively ommiscient beings', i.e. beings who know all the strict consequences of what they know, and also that they know that every other being is deductively ommiscient.

The theory of semantic information would be vastly improved by a diminution in the role of "ideal receivers"; for no human or animal receiver can in any way approximate to an ideal receiver.

As with the theory of logical content, so with the theories of semantic information measure; the theories can be substantially improved and major anomalies removed simply by setting the theories on a more adequate base, by replacing the underlying classical basis consisting of state descriptions or possible worlds by a new semantical base which includes further classically-neglected worlds, notably inconsistent and radically-incomplete worlds. Such a new base ultralogic provides. We have already seen the dramatic, and beneficial, effect of the change of basis in the case of explications of content. Let us consider now a similar revamping of the Carnap-Bar-Hillel proposed quantitative explications of semantic information. These revampings also foreshadow moves to come: for practically the whole of Carnap's semantical work, including the theory of logical probability and its ramifications, can be similarly refounded, on ultralogical foundations.

Carnap and Bar-Hillel offer two explications of amount of information, both of them important. Let us examine, in detail, an ultralogical resetting of the first explication, that of content measure.

In order to set out this account and to pave the way for ultramodal probability theory, the theory of semantic measures, on deductive situations, has first to be sketched. Previously semantical measure theories have all been based essentially on the theory of state descriptions or possible worlds. Thus all the measures and relations introduced have been modal, in that strict or provable equivalents are intersubstitutable everywhere for one another. This is already a mistake: a genuinely propositional measure should only allow generally the intersubstitutivity of propositional identicals. Many other deficiencies in the standard approaches, leading to philosophical puzzles, result from the use of possible worlds. However the usual theory can be generalized, and can be based on improved semantical accounts of propositional or content identity. Whatever the correct semantical theory of propositional identity is - and it is certainly not the modal theory or the purely sentential theory - propositional measure theory can be based on it. It has been argued elsewhere (especially in [6]) that one correct account of propositional identity is given, at the first degree (i.e. where no nesting of entailments or identities occur), by the theory of deductive situations, that propositions are given by logical content, or, what comes to the same, by ranges, and that truth functional assertions A and B are propositionally identical iff they hold in the same deductive situations. (When higher-degree expressions are introduced, a more sophisticated account is needed; see again [6]. But all the measure theory that follows will be confined to the first degree.)

Propositional measure theory simply extends the measure theory of Carnap [38]. In other words, the main definitions used are the same as in the case of measures based on model sets or state descriptions. Because however the semantical basis is different one gets different and more satisfactory results - more general in application, and less paradoxical.

Consider, as has become customary, the finite case, where all connectives are truth functional and quantified wff are expanded truth-functionally. The method in the infinite case, obtained in Carnap's way by taking limits, goes over largely intact. The finite case is typically obtained by supposing that there are k individual constants and m predicates, so that there are finitely many elementary assertions, e.g. k X m if all the predicates are one-place, as is often assumed, to avoid problems of dependence that the modal treatment runs into, but which the ultramodal treatment avoids. At any rate it is supposed in the finite case that there just n initial assertions or wff, for some n. These generate 4^n situations $a_1,\ldots,a_4n = a_m$. Where $p_1,\ldots,p_n$ represent the n wff, the 4^n situations may be listed by constructing a holding (or truth-table type) assignment for p_1, $\sim p_1$, p_2, $\sim p_2$, $\cdots$, p_n, $\sim p_n$. For example, where n=2, we set out the holding assignment (which treats p_i and $\sim p_i$ as independent variables), writing 1 where p_i holds and 0 where it does not;

p_1	$\sim p_1$	p_2	$\sim p_2$
1	1	1	1
1	1	1	0
1	1	0	1
1	1	0	0
1	0	1	1
1	0	1	0
1	0	0	1
1	0	0	0
0	1	1	1
0	1	1	0
0	1	0	1
0	1	0	0
0	0	1	0
0	0	0	1
0	0	0	0

Corresponding situations, in terms of what holds in them

p_1, $\sim p_1$, p_2, $\sim p_2$

p_1, $\sim p_1$, p_2

p_1, $\sim p_1$, $\sim p_2$

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Λ

Where n=1 there are 4 situations:

a_1	p_1, $\sim p_1$
a_2	p_1
a_3	$\sim p_1$
a_4	Λ , i.e. neither p nor $\sim$p holds in a_4.

Next measures are assigned to situations. The question arises: why start with measures on situations: why not simply begin with measures assigned to propositions and work back to measures on situations? Jeffrey [37] shows that this procedure would be less general. One cannot work back uniquely from measures for propositions

to measures for situations, so one would have to assign measures to situations as well!

A propositional measure function m simply assigns to each situation a in class of situations K an (extended) real number m(a). Function m is usually constrained by further conditions, e.g. in probability and information theory it is typically required that:

a. $m(a) \geqslant 0$, for each $a \in K$, and

b. $\sum_{a \in K} m(a) = 1$.

Such a measure function is called a normal (regular in Carnapian probability theory) m-function, because the measure is normalised by the requirements. To require, however, as Carnap's work suggests, that the measure is zero at most on worlds which are not possible ones is a deficiency: one wants to be able to restrict assessments to natural situations for example.

Measures for propositions are defined in terms of measures of the situations in which they hold (or in which they fail to hold). Since the proposition that A is given by the class of situations where A holds such a connexion is to be expected: for then

$$m(A) = m'(\S A) = m' \ominus\{a:I(A, a) = 1\}$$
$$= f\ \{m(a):I(A, a) = 1\}\ ,$$

assuming that the situations count just because of their measures. (The matter is in general not quite so simple as this because the measure may depend on other situational measures as well, e.g. probability weighted measures for desirability measures.)

Two connected issues which arise are these: What function of the range of A is m(A)?, and how much variation in the function can be taken up in the latitude allowed in assigning measures to situations? Carnap argues, in the case of probability measures, that the measure of A is the sum of the m-values of situations in the range of A; i.e. he argues ([38], p.279) for the condition:

c. $m(A) = \sum_{a \in \{c:I(A, c) = 1\}} m(a)$,

This assignment has been taken over and defended by later writers (e.g. Jeffrey [37]).[21] Call such an assessment rule the normal rule, and call normal measures those where the normal rule applies to normal m- functions. The ultralogical theory of normal measures will be elaborated and compared with Carnap's modal theory when we come to probability logic (in the next section).

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[21] The rule in fact follows upon assumption of certain conditions of adequacy, as Kemeny shows in his generalisation of Carnap's theory: see [40]. It is not at present clear whether or not this procedure can be ultramodally extended, since the argument given makes heavy use of the modal fact that m(C) = 1 for every tautology C.

Normal measures do not suffice for all the many purposes[22] for which semantical measure theory is wanted. For example, for semantical preference and decision theory a more sophisticated measure seems to be needed. But for semantic information theory normal measures appear to suffice, though the scale, fixed at the bottom at states of no information by zero, has an arbitrarily chosen upper bound.

Given a logical measure m on deductive situations, the first explication of amount of information, that of content measure, is as follows:-

$$\text{cont}(A) = \sum_{a \in \{c:I(A,\, c) \neq 1\}} m(a) \ ,$$

$$\text{i.e.} \quad \sum_{a \in c(A)} m(a) \ ,$$

i.e. the content measure of A is the sum of the measures on the content of A. Given modal measures, i.e. that all logical situations are possible worlds or determined by state descriptions, the Carnap-Bar-Hillel definition

$$\text{cont}(A) = m(\sim A)$$

follows. For

$$m(\sim A) = \sum_{a \in \{c:I(\sim A,\, c) = 1\}} m(a)$$

$$= \sum_{a \in \{c:I(A,\, c) \neq 1\}} m(a)$$

$$= \text{cont}(A) \ .$$

But with the full range of logical situations admitted, this undesirable reduction does not ensue. To show the merit of the ultramodal theory, it is now compared with the Carnap-Bar-Hillel theory. Theses of the latter modal theory are listed and their status as ultralogical semantical theses (shown by $\models$) or rejections ($\not\models$) indicated.

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[22] Modal measures from the three following groups have been studied, more or less extensively:

(1) Probability, (credibility), confirmation, confidence.

(2) Utility, preferability, desirability, and other valuation and choice measures (e.g. goodness).

(3) Content, informativeness.

Clearly the list could be extended. Thus Rescher has added:

(4) Cost, feasibility.

He introduces C(p), the cost of bringing it about (assuring) that p, i.e. the price that must be paid to guarantee the truth of p. According to Rescher the logic of cost is isomorphic with the (modal) logic of information.

Ultralogical accounts of many of these notions (ultramodal preference theory, decision theory, measure theory, confirmation theory, etc.) are attempted in [29].

$\models 1 \geqslant \text{cont}(A) \geqslant 0$; $\not\models \text{cont}(A) = 1 - m(A)$; $\not\models \text{cont}(\sim A) = m(A)$.

$\not\models \text{cont}(A) = 0$ iff A is necessary (L-true); $\not\models \text{cont}(A) = 1$ iff A is impossible (L-false).

Thus necessary propositions do not carry no measure of information, and impossible propositions do not convey total information.

$\models$ if $A \Rightarrow B$ then $\text{cont}(A) \geqslant \text{cont}(B)$; $\not\models$ if $A \not\Rightarrow B$ then $\text{cont}(A) \geqslant \text{cont}(B)$.

$\models$ if $A \Leftrightarrow B$ then $\text{cont}(A) = \text{cont}(B)$; $\not\models$ if $A \not\Leftrightarrow B$ then $\text{cont}(A) = \text{cont}(B)$.

$\models \text{cont}(A \,\&\, B) \geqslant \text{cont}(A) \geqslant \text{cont}(A \vee B)$.

$\models \text{cont}(A \vee B) = \text{cont}(A) + \text{cont}(B) - \text{cont}(A \,\&\, B)$.

$\models \text{cont}(A \vee B) = \text{cont}(A) + \text{cont}(B) - 1$, iff A excludes B, where A excludes B iff $c(A) \cup c(B) = \mathcal{V}$, i.e. $r(A) \cap r(B) = \Lambda$, i.e. the ranges of A and B are disjoint. The last result follows because

$$\text{cont}(A \,\&\, B) = 1 \text{ iff } 1 = \sum_{a \in \{c : I(A, c) \neq 1 \vee I(B, c) \neq 1\}} m(a)$$

$$\text{iff } 1 = \sum_{a \in c(A) \cup c(B)} m(a)$$

iff A excludes B.

$\models \text{cont}(A \,\&\, B) = \text{cont}(A) + \text{cont}(B) - \text{cont}(A \vee B)$; $\models \text{cont}(A \,\&\, B) = \text{cont}(A) + \text{cont}(B)$ iff A disjoins B,

where A disjoins B iff $c(A) \cap c(B) = \Lambda$. (It is here assumed that $\sum_{\Lambda} m(a) = 0$.)

$\not\models \text{cont}(\sim A) = 1 - \text{cont}(A)$; this rejection is central to the ultramodal theory; In contrast, where m is a modal measure, $\models \text{cont}(\sim A) + \text{cont}(A) = 1$, and conversely. For, for arbitrary A, $\text{cont}(\sim A) + \text{cont}(A) = 1$

iff $c(A) \cup c(\sim A) = \mathcal{V}$ (i.e. the class K of all situations)

iff $\{a : I(A, a) \neq 1\} \cup \{a : I(\sim A, a) \neq 1\} = K$

iff $\{a : I(A, a) \neq 1 \vee I(\sim A, a) \neq 1\} = K$

iff $\{a : I(A \,\&\, \sim A, a) \neq 1\} = K$

iff for every a, $I(A \,\&\, \sim A, a) \neq 1$, i.e. just the modal requirement.

The relative amount of information of B with respect to A, $\text{cont}(B/A)$, is defined, in terms of the absolute amount, in the following way by Carnap and Bar-Hillel:

$$\text{cont}(B/A) = \text{cont}(A \,\&\, B) - \text{cont}(A).$$

$\models$ cont(B/A) = cont(B) - cont(A $\vee$ B); for cont(A & B) - cont(A) = cont(A) + cont(B) - cont(A $\vee$ B) - cont(A).

$\models$ cont(B/A) = cont(B) iff A disjoins B.

$\models$ cont(B/A) = cont(A $\supset$ B), provided m is a modal measure.

$\not\models$ cont(B/A) = cont(A $\supset$ B), i.e. the relative content-measure of B given A is not the same as the absolute content-measure of the material implication A $\supset$ B. In defence of this principle - for modal measures - Carnap and Bar-Hillel try to make out that

> If an 'ideal' receiver possesses the knowledge i and then acquires the knowledge j, his possession of information is only increased in the same amount as if i $\supset$ j were added instead of j. This is, indeed, highly plausible since j is a logical consequence of the sentences i and i $\supset$ j, and an 'ideal' receiver, by definition, is able to draw such consequences instantaneously.

This is patent sophistry. Though $\models$ cont(A $\supset$ B) = cont($\sim$A $\vee$ B), Carnap and Bar-Hillel have to rely on the paradoxical proposition that cont(B) = cont(A $\vee$ B) + cont($\sim$A $\vee$ B) and on the modal thesis that if A $\not\rightarrow$ B then cont(B/A) = 0 - both rejected on ultramodal assumptions. All that is true is:

$\models$ if A $\Rightarrow$ B, cont(B/A) = 0. For if A $\Rightarrow$ B is a thesis, then so is A & B $\Leftrightarrow$ A; so cont(A & B) = cont(A), whence cont(B/A) = 0.

It is worth observing that corresponding transmission principles for epistemic functors such as knowledge - as distinct from logical notions like information - fail. Finally, $\models$ cont(B/A) $\leqslant$ cont(B).

The specific results adduced by Carnap and Bar-Hillel which depend on defining content in terms of a proper measure function fail because they depend essentially on the mistaken principle that tautologies have no content. Analogues of their results could, however, be derived using cont(A/t), with t the sentential constant previously introduced, under the (false) hypothesis that cont(t) = 0.

Other major theses of the Carnap-Bar-Hillel theory also require special assumptions. The following interesting result fails where m is not a proper measure or not a modal measure:

$\models$ where m is a proper modal measure, and B_i and B_j are basic sentences with different primitive predicates,

$$\text{cont}(B_j/B_i) = 1/4 = 1/2\ \text{cont}(B_i).$$

The background assumptions required for this significant proposition are worth elaborating. The result depends firstly on the following conditions on proper measures:

d. If B_j is formed from B_i by replacing any of the primitive predicates of B_i by their negations (omitting double negation signs), then $m(B_j) = m(B_i)$ (i.e. each primitive property is treated on a par with its complement).

e. If A and B have no primitive predicates in common, then m(A & B) = m(A) X m(B).

Then $\models$ for any B, $m_{pm}(B) = 1/2$, where m_{pm} is a proper modal measure. For since

$$\vdash B \vee \sim B,\ 1 = m_{pm}(B \vee \sim B) = m_{pm}(B) + m_{pm}(\sim B)$$
$$= 2m_{pm}(B) \quad \text{by d.}$$

There are three further preliminary results:-

$\models$ for any conjunction C_n of n basic sentences with n distinct predicates, $m_{pm}(C_n) = (1/2)^n$, by applying e.

$\models$ for any basic sentence B, $cont_{pm}(B) = 1/2$.

$\models$ for any conjunction C_n of n basic sentences with n distinct primitive predicates, $cont_{pm}(C_n) = 1 - (1/2)^n$.

Assembling these preliminaries it follows:

$$\models cont(B_j/B_i) = cont(B_j \,\&\, B_i) - cont(B_i)$$
$$= (1 - 1/4) - 1/2 = 1/4$$
$$= 1/2\ cont_{pm}(B_i).$$

Carnap and Bar-Hillel comment:

> According to this theorem, if an 'ideal' receiver with no previous knowledge receives a sequence of n basic sentences with n different primitive predicates, the amount of information he gets from the first sentence is 1/2, from the second only 1/4, from the third 1/8, from each only half as much as from the preceding one. And this will be the case despite the fact that these basic sentences are independent from each other not only deductively but also inductively. One has the feeling that under such conditions the amount of information carried by each sentence should not depend upon its being preceded by another of its kind.

Call this the CBH paradox; for it is paradoxical that the amount of information of independent sentences should diminish in this way. The paradox appears to destroy the Carnap and Bar-Hillel theory. The paradox arises from a conflict between requirements on modal measures on the one hand and on proper measures on the other; for the first requirements make the measure of ~fa dependent on the measure of fa (as 1 - measure of fa) whereas the second requirements, because of d and e, treat negated basic sentences as virtually independent.

Carnap and Bar-Hillel propose to "resolve" the conflict of intuitions engendered by the paradox - which they surprisingly try to ascribe to a conflict as to which condition to impose on the additivity requirement cont(A & B) = cont(A) + cont(B) - by supposing that there is not one explication of "amount of semantic information"

but at least two.[23] This is however but a transparent pretext for introducing a measure resembling the Shannon-Wiener measure of statistical information theory.

The new proposal is as follows:

$$\inf(A) = \text{Log}\ \frac{1}{1 - \text{cont}(A)} \quad ,$$

where 'Log' is short for 'logarithm to the base 2'. This is equivalent for modal measures to:

$$\models \inf(A) = \text{Log}\ \frac{1}{m(A)} = -\text{Log}\ m(A) \quad ,$$

and is analogous to the customary definition of amount of information in communication theory, except that, in place of the (probability) measure m, statistical probability is used in the communication theory definition. A second explication of amount of information by a semantical measure then simply takes over this connexion to define:

$\inf(A) = -\text{Log}\ m(A)$. So $\models m(A) = 2 - \inf(A)$.

Then a modified version of the Carnap-Bar-Hillel theory results with differences appearing in the same set of places as before: e.g.

$\models 0 \leqslant \inf(A) \leqslant \infty$; $\not\models \inf(A) = 0$ iff A is necessary; $\not\models \inf(A) = \infty$ iff A is impossible.

$\models$ if $A \rightarrow B$ then $\inf(A) \geqslant \inf(B)$; $\not\models$ if $A \not\rightarrow B$ then $\inf(A) \geqslant \inf(B)$.

Additivity holds in the Carnap-Bar-Hillel form:

$\models \inf(A \,\&\, B) = \inf(A) + \inf(B)$ iff A and B are inductively independent, i.e. $m(A \,\&\, B) = m(A) \times m(B)$.

Proof:
$$\begin{aligned} \inf(A) + \inf(B) &= -\text{Log}\ m(A) + -\text{Log}\ m(B) \\ &= \text{Log}\ 2^{\inf(A)} + \text{Log}\ 2^{\inf(B)} \\ &= -\text{Log}\ (2^{-\inf(A)} \times 2^{-\inf(B)}) \\ &= -\text{Log}\ (m(A) \times m(B)) \\ &= -\text{Log}\ m(A \,\&\, B) \quad . \end{aligned}$$

More generally, results independent of negation and paradoxes hold in the classical form.

§12. Ultramodal probability theory. If semantical information theory can be more satisfactorily reworked ultramodally, should not the same apply to logical probability theory, especially as probabilification is a sort of information-conveying relation? The answer is, of course, yes,

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[23] The method of finding ambiguity in the explicandum or in ordinary notions is a favourite formalist dodge in the face of paradox. Mostly, however the paradoxes generated are engendered by the inadequacy of the underlying formalism to the explication required - inadequacy commonly attributable to conformity to extensional or modal constraints.

and that, once again, several anomalies can be removed from classical probability theory - which is thoroughly modal - by ultramodal remodelling. But the case for the remodelling is different, and really needs to be developed afresh.

There are two familiar theses, both I believe correct, from which a new beginning can be made, namely

(I) There is a logical relation between premiss and conclusion of an argument, or antecedent and consequent of an implication, of extent of conclusiveness, or, as I shall say, following more recent literature, a relation of partial entailment. If there is such a relation it can of course be extended, quantified and metrised.

(II) Just such a logical relation provides a (sometimes, it is claimed, the) major explication of the relation of logical probability and also (in one sense again) of confirmation.

These theses have been extensively argued for in the literature, notably by Keynes [39] and Johnson [41]; the theses remain very much live issues and they have been defended recently, e.g. by Stove [43].

Given these theses the next question is as to the logical properties of the relation, which can be represented in a standard metrical way as P(h, e) = r, i.e. e entails h to degree r; that is to say, given the theses, the logical probability of h on or given e is r, where r is some real number, conventionally restricted to the closed interval [0,1]. Now degree of conclusiveness of an argument coincides, in the maximal case, i.e. where P(h, e) = 1, with the relation of valid argument, that is, on both classical and relevant grounds, it coincides with e's entailing h. That is, maximal conclusiveness of antecedent of an implication for consequence just is entailment. But entailment should be explicated, as before, ultramodally, not as a strict implication relation as the classical theory of probability would have. In short, logical probability, explicated in terms of degrees of entailment, properly requires an ultramodal analysis. Fortunately a semantics for such a theory is virtually at hand; at least for first degree wff, Carnap's modal semantics for logical probability straightforwardly generalises to an ultramodal semantics based on deductive situations.

The argument for a new and different probability logic does not rest merely on the fact that in a limiting case maximal partial entailment, which just is entailment, should be a good entailment, a proper sufficiency relation. Rather similar points apply in the case of less than maximal partial entailment as apply for entailment itself. Where A partially entails B what should be captured is whether A on its own partially ensures B, not whether A together with all logical truths or necessary propositions partially guarantees either B or some logical falsehood - which is what the modal analysis conflates with partial entailment. For a satisfactory account all these non-contingent propositions which the modal account suppresses should be discarded. And this can be done semantically exactly as in the analysis of full entailment, by considering situations where necessary propositions fail and also situations where logical falsehoods hold. Thus the measure in terms of which degrees of partial entailment are

defined should extend out over incomplete and inconsistent deductive situations.

Accordingly for a metrical analysis of partial entailment, just as for semantic information, measures are taken over all deductive situations. As there too, it can safely be assumed that measures are normal. For probability logic some of the theory of normal measures is needed. At the same time as setting out valid and invalid assertions on the ultramodal theory, it is worth comparing the theory with Carnap's modal account from which it derives. Recall, to begin, the conditions on normality, especially

c. $m(A) = \sum_{a \in r(A)} m(a)$,

where the range of A, r(A), is the class of deductive situations where A holds, i.e. $\{c \in K: I(A, c) = 1\}$.

$\models$ if A entails B then $m(A) \leqslant m(B)$; $\models\!\!\!|$ if A strictly implies B, then $m(A) \leqslant m(B)$.

$\models$ if $A \leftrightarrow B$ (A coentails B) then $m(A) = m(B)$; $\models\!\!\!|$ if A is strictly equivalent to B, then $m(A) = m(B)$.

$\models m(A \,\&\, B) \leqslant m(A) \leqslant m(A \vee B)$.

Proof: From the entailment result above. Alternatively, $m(A \,\&\, B) = \sum_{a \in \{c: I(A \,\&\, B, c) = 1\}} m(a) = \sum_{\{a: I(A, a) = 1\} \cap \{a: I(B, a) = 1\}} m(a) \leqslant \sum_{\{a: I(A, a) = 1\}} m(a)$.

$\models m(A \vee B) = m(A) + m(B) - m(A \,\&\, B)$.

Proof: $m(A \vee B) + m(A \,\&\, B) = \sum_{r(A) \cup r(B)} m(a) + \sum_{r(A) \cap r(B)} m(a) =$

$\sum_{r(A)} m(a) + \sum_{r(B)} m(a) - \sum_{r(A) \cap r(B)} m(a) + \sum_{r(A) \cap r(B)} m(a) = \sum_{r(A)} m(a)$

$+ \sum_{r(B)} m(a) = m(A) + m(B)$.

$\models m(A \,\&\, B) = m(A) + m(B) - m(A \vee B)$; $\models m(A \vee B) = m(A) + m(B)$ iff $m(A \,\&\, B) = 0$; $\models m(A \,\&\, B) = m(A) + m(B) - 1$ iff $m(A \vee B) = 1$; $\models m(A \,\&\, B) \leqslant m(A) + m(B)$.

$\models$ if $m(A \,\&\, B) = 1$, then $m(A) = 1 = m(B)$; for $1 \geqslant m(A) \geqslant m(A \,\&\, B)$.

$\models$ if $m(A) = 0$, then $m(A \,\&\, B) = 0$; and $\models$ if $m(A) = 0 = m(B)$ then $m(A \vee B) = 0$.

Proof: if $m(A) = 0$ then $m(A \,\&\, B) = 0$, so $m(A \vee B) = m(A) + m(B)$.

$\models$ if $m(A) = 1 = m(B)$ then $m(A \,\&\, B) = 1$.

$\models m(A_1 \vee \ldots \vee A_n) = \sum_{i=1}^{n} m(A_i)$, where $m(A_\alpha \,\&\, A_\beta) = 0$ for every α, β in (1, n).

But there is no need to go about proving very many of these results directly. For every modal result which avoids use of the negation rule carries over intact to the ultramodal theory, since all positive requirements, those for conjunction, disjunction and the quantifiers, coincide. (Thus too ultramodal probability theory represents a less drastic departure from the classical theory than the orthological probability logic exponents of orthological quantum logic are strictly committed to, since this theory must diverge also on disjunction and particularity.) A (normal) modal measure is a normal measure m such that $m(a) = 0$ iff a is not a consistent and complete world, i.e. $m(a) > 0$ iff a is a (complete) possible world.

The theorems on modal measures are exactly those established by Carnap [38] for his measures. Any positive theorem on modal measures - more generally of modal probability theory - is a theorem on ultramodal measures - more generally of ultramodal probability theory. Algebraically this comes out very clearly: while classical probability theory amounts to a measure theory on Boolean algebras, ultramodal probability theory amounts to such a measure theory on De Morgan lattices.

The fundamental difference between ultramodal measures and classical ones shows up with negation. Though $\models m(A) + m(\sim A) \leqslant 1$, $\not\models m(\sim A) = 1 - m(A)$. $m(\sim A)$ and $m(A)$ may behave much more independently, a most important feature when it comes to applications. Likewise classical expansions, analogous to implication paradoxes, are rejected: $\not\models m(A) = m(A \,\&\, \sim B) + m(A \,\&\, B)$. The difference over negation spills over into differences as to non-contingent assertions. Though $\models 0 \leqslant m(A) \leqslant 1$ of course, $\not\models 0 < m(A) < 1$ iff A is contingent; $\not\models m(A) = 1$ where A is valid (more sweepingly, iff A is logically necessary); and $\not\models m(A) = 0$ where A is logically false. By making use of the constant t introduced earlier the classical theory can be represented ultramodally: the assumption is always: $m(t) = 1$ & $m(f) = 0$, where $f =_{Df} \sim t$. Then as $t \rightarrow A$ where A is a theorem, if $m(t) = 1$, $m(A) = 1$; so $m(A) = 1$ given $m(t) = 1$. Similarly as $\sim A \rightarrow \sim t$ when A is a theorem, $m(\sim A) \leqslant m(\sim t) = 0$; so $m(\sim A) = 0$ given $m(\sim t) = 0$. Hence, under the assumption, $m(A \,\&\, \sim A) = 0$, so too $m(A) + m(\sim A) = 1$, i.e. $m(\sim A) = 1 - m(A)$.

A partial entailment ratio is now defined in terms of semantical measures as follows: c(A, B) is defined as the sum of the measures of B-situations where A holds divided by the sum of measures of B-situations. But the measures of B-situations, i.e. situations where B holds, where A also holds, just are the measures of (A & B)-situations. Thus

$$c(A, B) =_{Df} m(A \,\&\, B)/m(B).$$

To complete the definition the case where $m(B) = 0$ has to be dealt with. It should be noted that this case will not arise unless extra conditions, such as the assumption $m(B) = 0$, are imposed on the modelling. Let us stipulate that where $m(B) = 0$, $c(A, B) = 1$, though this raises, as Carnap points out ([38], p. 296) difficulties in the infinite case on the limit approach. (If we had chosen a more suitable range for values of the measure function, e.g. the interval $[0, +\infty]$ instead of the interval $[0, 1]$, dictated by the frequency theory of probability, further stipulation would have been unnecessary.)

Where m is a normal measure, the corresponding confirmation function c is called, following Carnap, a regular confirmation function, and if m is modal, c is also modal. All the results which follow in the sketch of ultramodal probability theory, and comparison with Carnap's theory, are established for regular functions in the finite case. To facilitate the comparison with Carnap, his notation is adopted, 'h' for hypothesis statement, 'e' for evidence, etc.

$\models 0 \leqslant c(h, e) \leqslant 1$; $\not\models$ if $\vdash e \supset h$ then $c(h, e) = 1$; $\not\models$ if $e \dashv h$ then $c(h, e) = 1$.

$\models$ If $e \rightarrow h$, then $c(h, e) = 1$.

Proof: If $e \rightarrow h$ then $e \,\&\, h \leftrightarrow e$, so $m(e) = m(e \,\&\, h)$.

$\not\models$ If h is L-true (necessary), $c(h, e) = 1$; $\not\models$ $c(t, e) = 1$; $\not\models$ if $\vdash e \supset \sim h$, then $c(h, e) = 0$; $\not\models$ if h is L-false (impossible), $c(h, e) = 0$; $\not\models$ $c(\sim t, e) = 0$.

$\not\models$ If $\vdash e_1 \equiv e_2$ $(e_1 \leftrightarrows e_2)$ then $c(h, e_1) = c(h, e_2)$; $\models$ if $e_1 \leftrightarrow e_2$ then $c(h, e_1) = c(h, e_2)$, identical evidence.

$\not\models$ If $\vdash h_1 \equiv h_2$ $(h_1 \leftrightarrows h_2)$ then $c(h_1, e) = c(h_2, e)$; $\models$ if $h_1 \leftrightarrow h_2$ then $c(h_1, e) = c(h_2, e)$, (propositionally) identical hypotheses.

In short, substitutivity conditions are of entailmental, not of modal, strength. This is what distinguishes the theory being developed as an ultramodal one. It should be evident that as partial entailment claims look out over all deductive situations, nothing short of entailmental coincidence is an adequate basis for intersubstitutivity: strict coimplication certainly should not be.

$\models c(h \vee i, e) = c(h, e) + c(i, e) - c(h \,\&\, i, e)$, general addition theorem.

Proof: $\underline{m((h \vee i) \,\&\, e)} = \underline{m(h \,\&\, e \vee i \,\&\, e)}$
$= \underline{m(h \,\&\, e)} + \underline{m(i \,\&\, e)} - \underline{m(h \,\&\, i \,\&\, e)}$,

whence the result follows. The divisor is in each case m(e). A corollary is the special addition theorem:

$\models c(h \vee i, e) = c(h, e) + c(i, e)$, where $c(h \,\&\, i, e) = 0$.

$\models c(h \,\&\, i, e) = c(h, e) \times c(i, e \,\&\, h)$
$= c(i, e) \times c(h, e \,\&\, i)$, general multiplication theorem.

Proof: $\dfrac{m(h \,\&\, i \,\&\, e)}{m(e)} = \dfrac{m(h \,\&\, e)}{m(e)} \times \dfrac{m(i \,\&\, e \,\&\, h)}{m(e \,\&\, h}$

$\models$ If $e \,\&\, h \rightarrow j$ then $c(h \,\&\, j, e) = c(h, e)$.

Proof: $e \,\&\, h \,\&\, j \leftrightarrow e \,\&\, h$, so $m(e \,\&\, h \,\&\, j) = m(e \,\&\, h)$, whence the result follows.

$\models c(h \,\&\, e, e) = c(h, e)$.

$\models c(h, e \,\&\, i) = \dfrac{c(h, e) \times c(i, e \,\&\, h)}{c(i, e)}$, general division theorem.

Proof: By the multiplication theorem:

$c(h, e) \times c(i, e \,\&\, h) = c(i, e) \times c(h, e \,\&\, i)$.

As expected then, positive theorems of classical probability theory emerge unscathed. The differences emerge with negation and negation-defined notions such as exclusion. The important difference, from the point of view of the classical axiomatisation of probability theory, can be located in the rejection: $\not\models c(\sim h, e) = 1 - c(h, e)$. And this has the important upshot ultramodally that both $c(h, e)$ and $c(\sim h, e)$ may have low probability values, and that both may have high values, as may happen in the case of so-called "inductive inconsistencies".

The essential result of this independence of values, of this new liberality, is a theory of vastly wider applicability than the modal theory. Just how will be no more than indicated here.

Firstly, there are a great many cases where, on given evidence, which may be slight or even zero, neither a hypothesis h nor its negation ~h has much, or even slight, probability. On the total evidence available to the ancient Sumerian neither the big bang hypothesis nor its negative had other than negligible probability.

Secondly, there are many cases where on the given evidence both a hypothesis and its negation have a high probability. Suppose the evidence e consists of the corpus of Newtonian physics together with successful empirical observation at a time shortly after the Michaelson-Morley experiment, and let the hypothesis h be that the light beams compared by the experiment take equal time for their respective journeys. Then $c(h, e)$ is large because of the experimental result, and $c(\sim h, e)$ is also large because of the predictions of an ether wind, and its effect on light beams, from the Newtonian corpus. It might be objected that the evidence e is physically inconsistent evidence. True, but nothing excludes such evidence. In fact, ultramodally the essential claim under discussion can be readily proved. Let $e = p \,\&\, \sim p$ and $h = p$. Then $c(h, e) = c(\sim h, e) = 1$.

Thirdly, then, the ultramodal theory can accommodate logically inconsistent evidence, which the classical theory has to rule out, on pain of inconsistency otherwise. Thus Carnap shows ([38], p. 341) that Jeffrey's axiom system, which fails to exclude self-contradictory evidence, is inconsistent in a damaging way.

The argument may be redrafted as follows:- Since $p \,\&\, \sim p \rightarrow p$ and $p \,\&\, \sim p \rightarrow \sim p$, $c(p, p \,\&\, \sim p) = 1 = c(\sim p, p \,\&\, \sim p) = 1 - c(p, p \,\&\, \sim p)$, by modal muddle. Thus $1 = c(p, p \,\&\, \sim p) = 0$, so $1 = 0$. But, contrary to what Carnap suggests, the restriction to non-self-contradictory evidence is a severe restriction on the scope of the modal theory. It rules out the dialectical application of the theory almost immediately (not that that would worry Carnap). Worse, it interferes seriously with the underlying idea of the probability relation as one of degree of conclusiveness of arguments; for of course arguments can have

inconsistent premisses, and a general theory should allow for such premisses. We find a curious tension in Keynes on this point. Having contended that one must be prepared to consider probability relations between any pairs of sets of propositions, he has immediately to qualify his appealing thesis - on the grounds of convenience!

> Probability is concerned with arguments, that is to say, with the "bearing" of one set of propositions upon another set. If we are to deal formally with a generalised treatment of this subject, we must be prepared to consider relations of probability between any pair of sets of propositions, and not only between sets which are actually the subject of knowledge. But we soon find that some limitation must be put on the character of sets of propositions which we can consider as the hypothetical subject of an argument, namely, that they must be possible subjects of knowledge. We cannot, that is to say, conveniently apply our theorems to premisses which are self-contradictory and formally inconsistent with themselves.

But Keynes's generality argument ([39], p. 111) surely applies generally. It is not a matter of convenience but a limitation on classical and modal theories that they cannot be applied generally. The ultramodal theory removes that limitation.

There remain two further apparently substantial advantages of the ultramodal theory that I wish to advance with a little more caution, until the theory is more fully worked out. Firstly, the theory enables one to avoid standard objections to detachment in the case of statistical inference. Consider the case of statistical syllogisms, that is arguments of the form:

a is F:
The proportion of F's that are G is q.
Therefore, it is probable, to degree q, that a is G.

The standard objection to accepting any arguments of this form is clearly set out by Hempel ([43], p. 131) who contends that such

> syllogisms generate inductive inconsistencies in the following sense: For an argument with true premisses that has the form of a statistical syllogism, there exists, in general, a rival argument of the same form, again with true premisses, whose conclusion is logically incompatible with that of the first premiss.

But the conclusions of the rival arguments are not logically incompatible, and the idea that they are incompatible derives from the modal assumption that it cannot be the case that both p and ~p have a high probability. For the conclusions are of the form: P_q G(a) and P_q ~G(a) and these are not incompatible without the mistaken consistency assumption P_q ~G(a) → ~P_q G(a), deriving from classical negation assumptions. It is tempting to toy with the idea that ultramodally detachment in the case of certain statistical inferences is in order, and that the crippling requirement of total evidence that the modal theory has to impose can be avoided. (This does not mean that that the total evidence requirement does not have

an important place in decision theory.) Naturally the reinstatement of detachment from statistical syllogisms would give a considerable fillip to the business of accounting for and justifying inductive reasoning and procedures.

The second advantage of the ultramodal theory, that I tentatively advance for your consideration, would likewise provide a fillip to probability theory, since it would enable initial probabilities to be assigned to confirmation relations. The idea is, that the theory will allow for the adoption of a principle related to the principle of indifference of classical probability theory, a principle to the effect that where there is no evidence δ as to a hypothesis h or its negation ~h then $c(h, \delta) = c(\sim h, \delta) = 0$. The same should apply where the evidence is irrelevant; but a proper discussion of this leads away into the important question of relevance under the ultramodal theory. Rather than follow that line of development let us turn to the objection that though the ultramodal theory may, perhaps, have some advantages, it sacrifices central and crucial parts of the classical and modal theories which turn on negation features.

To simulate leading classical results, such as Bayes's theorem, which, as commonly formulated, make use of classical negation features, a beginning needs to be made on the ultramodal theory of conditions restricting situations. There are conditions imposed to limit the class of situations considered. There are two classes of conditions that are important for the philosophy of science envisaged ultramodally.

1. Conformity conditions. Thus, for example, a situation a conforms to a lawlike conditional if A then B if when $A \in a$ then $B \in a$. Physical laws or lawlike conditionals may be imposed in this way to provide a class of lawlike situations, for use, e.g., in the semantical assessment of counterfactual conditionals. On classical and modal accounts, however, conformity just collapses into the holding of a material conditional in a situation, thereby smudging a fundamental distinction.

2. Exclusion and exhaustion conditions. These take the following forms, for a given situation a: if $A \in a$ then $B \notin a$, if $A \notin a$ then $B \in a$, $A \in a$ iff $B \notin a$. Exclusion conditions are important in limiting the class of situations or cases to be considered in probability applications, as in Bayes's theorem. In this sort of case one statement is used to rule another out everywhere. Fortunately many of Carnap's definitions of L-concepts ([38], p. 83 ff.) are tailor-made for ultramodal purposes, among them that of L-exclusion, though they no longer always carry the intended modal interpretation, e.g. L-exclusion no longer serves as an explication of logical impossibility of joint truth. A excludes B iff $r(A) \cap r(B) = \Lambda$. Thence $\models$ A excludes B iff, for every a in K, if $I(A, a) = 1$ then $I(B, a) \neq 1$, i.e. canonically if $A \in a$ then $B \notin a$; but $\not\models$ A excludes B iff $\vdash \sim(A \,\&\, B)$. A class S of wff is exclusive in pairs if every wff of S excludes every other wff of S.

$\models m(A \vee B) = m(A) + m(B)$ where A excludes B.

$\models$ if A excludes B, $c(A \vee B, e) = c(A, e) + c(B, e)$, an alternative form of the special addition theorem. For if A excludes B,

$r(A \,\&\, B \,\&\, e) = \Lambda$; so $m(A \,\&\, B \,\&\, e) = \sum_{a \in r(A \,\&\, B \,\&\, e)} m(a) = \sum_{a \in \Lambda} m(a) = 0$.

Thus $c(A \,\&\, B, e) = 0$, and the result follows from the special addition theorem. This theorem also follows, in the same way, with the hypothesis that A excludes B with respect to e, i.e. A & e excludes B & e.

Special addition theorem for multiple disjunction. If $h_1,\ldots,h_n$ $(n \geqslant 2)$ are exclusive in pairs (with respect to e) then

$$c(h_1 \vee h_2 \vee \ldots \vee h_n, e) = \sum_{n=1}^{p} c(h_p, e).$$

Proof: From the preceding theorem by induction.

Preliminary theorem to Bayes's. Where $j = j_1 \vee \ldots \vee j_n$ (for $n \geqslant 2$),

(i) $e \,\&\, h \rightarrow j$, and

(ii) $j_1,\ldots,j_n$ are exclusive in pairs (w.r.t. e or h),

$$c(h, e) = \sum_{p=1}^{n} c(h \,\&\, j_p, e).$$

Proof: Since $e \,\&\, h \rightarrow j$, $e \,\&\, h \,\&\, j \leftrightarrow e \,\&\, h$. Thus applying previous results:

$$\begin{aligned} c(h, e) &= c(e \,\&\, h, e) \\ &= c(e \,\&\, h \,\&\, j, e) \\ &= c(h \,\&\, j, e) \\ &= c(h \,\&\,(j_1 \vee \ldots \vee j_n), e) \\ &= c(h \,\&\, j_1 \vee \ldots \vee h \,\&\, j_n, e) \quad \text{, by distributing} \\ &= \sum_{n=1}^{p} (h \,\&\, j_p, e) \quad \text{, by the preceding theorem.} \end{aligned}$$

Bayes's theorem. Where $c(i, e) > 0$ and $h_1,\ldots,h_n$ are such that

(i) $e \,\&\, i \rightarrow .\ h_1 \vee \ldots \vee h_n$, and

(ii) $h_1,\ldots,h_n$ are exclusive in pairs (w.r.t. e or i), then, where h is any one of $h_1,\ldots,h_n$,

a.(1). $$c(h, e \,\&\, i) = \frac{c(i \,\&\, h, e)}{\sum_{p=1}^{n} c(i \,\&\, h_p, e)}$$

(2). $$= \frac{c(h, e) \times c(i, e \,\&\, h)}{\sum_{p=1}^{n} [c(h_p, e) \times c(i, e \,\&\, h_p)]}.$$

b. Let $c(h_p, e)$ have the same value for every p (from 1 to n). Then

$$c(h, e \,\&\, i) = \frac{c(i, e \,\&\, h)}{\sum_{p=1}^{n} c(i, e \,\&\, h_p)}.$$

Proof: b is immediate from a(2). As to a,

$$c(h, e \,\&\, i) = \frac{c(h, e) \times c(i, e \,\&\, h)}{c(i, e)}$$, by the general division theorem

$$= \frac{c(h, e) \times c(i, e \,\&\, h)}{\sum_{1}^{n} c(i \,\&\, h_p, e)}$$, by the preliminary theorem

$$= \frac{c((h \,\&\, i), e)}{\sum_{1}^{n} c(i \,\&\, h_p, e)}$$. by the general multiplication theorem

$$= \frac{c(h, e) \times c(i, e \,\&\, h)}{\sum_{1}^{n} [c(h_p, e) \times c(i, e \,\&\, h_p)]}$$, by the general multiplication theorem.

The theory developed provides but a modest beginning to ultramodal metrical probability theory. (Qualitative and absolute probability theories will be presented in [29].) But several points should already be clear. Firstly, the theory can accommodate the results of classical logical probability theory by imposing conditions on situations. Secondly, however, the theory extends to cover significant cases where the classical theory fails, especially those where neither h nor ~h, or where both h and ~h are probable relative to evidence e. Thirdly, it should be fairly obvious that the theory can be recast algebraically by taking measures on De Morgan lattices. This leads to an interesting measure theory, beyond the reach of the present venture, which removes analogues of the paradoxes which appear, yet again, in classical measure theory, notably those concerning sets of measure zero.

§13. Ultramodal quantum logic. There is a basis for claiming that the bad effects of classical logic extend into science itself, at least in the case of quantum physics, and perhaps also in systematic taxonomy and rigid body dynamics. An outcome in the case of quantum physics - where proposals for quantum logics date back to the decade of the inauguration of quantum theory - is that many of those who have new-look logics have suggested that their sort of logic will work for quantum theory. (Hence the labyrinth of quantum logics that van Fraassen has observed.) Ultralogic may as well be on the act. There are several approaches that an ultramodalist may take with respect to alleged logical anomalies generated by quantum phenomena, in particular, a soft line which weakens the sentential logic by dropping or qualifying distribution, A & (B ∨ C) ↛. (A & B) ∨ (A & C), in line with the initial quantum logics, and a hard line which leaves the basic sentential logic unchanged.[24]

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[24] There are other lines to be tried as well, e.g. an ultramodal extension of van Fraassen's modal interpretation of quantum mechanics (see [44]).

Appropriate sentential logics for the soft line are developed and studied semantically in RLR, where it is argued that if classical logic is to be changed to accord with quantum features it can and should be adjusted to take account of other features it neglects (namely, those discussed in §2 above). But the soft line raises serious problems for the thesis that ultralogic is universal. For, firstly, relevant orthologics typically do not admit the rule (γ) of material detachment, and so interfere with the recovery of classical and modal logics. Secondly, the procedure of weakening the logic to deal with a class of quantum situations does not accord with, and indeed erodes, the thesis that ultralogic, with distribution in, is universal. Moreover the hard line has a reasonable chance of success.

The hard line is not hard like the classical stand, that the logic stands whatever the empirical data and that it is the physical theory that will have to be adjusted to take account of the curiosities and complications of quantum behaviour. For ultralogic has already taken substantial steps to modify classical logic theory, and these steps either accord with quantum logic criticisms of classical theory or else have already given away crucial parts of the classical theory on which the quantum criticism turns. As to the first point, the relevant critique of classical logic and its extensions joins with the quantum logical critique in rejecting the principle of Disjunctive Syllogism, and, more generally, in amending the classical account of negation. There is, then, substantial common ground between relevant and quantum critiques, with the result that most quantum arguments against classical logic do not apply against ultramodal logic.[25] The second point is this: the remaining quantum criticisms of classical theory, especially those that are supposed to tell against Distribution, all appear to turn on features not merely of classical quantification logic but essentially on negation features of classical probability logic - features which do not continue to hold in ultramodal probability theory.

The idea of a specific quantum logic to avoid the anomalies of orthodox quantum theory - proposed in the 1930's, not first, but most notoriously, by Birkhoff and von Neumann (see Jammer [53]) - has recently been revised and elaborated by Finkelstein (e.g. [45]), Putnam [46], and others. Finkelstein argues that <u>all</u> so-called "anomalies" of quantum theory, all the matters that are said to be meaningless under the orthodox interpretation, arise from the use of classical logic, which is quantum-mechanically false, and are removed by adoption of a non-standard logic. This is a controversial thesis, which has been savagely, if not very cogently, attacked; but we need not be concerned with its defence here. What is at issue is whether the non-standard logic can be the proposed universal logic, and so as to whether ultralogic can handle the cases Finkelstein and Putnam advance in favour of their thesis.

To confirm the first point - as to substantial common ground - let us consider Finkelstein's initial three cases of

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[25] Dunn persuaded me of the importance of this, of the merit of trying to do quantum theory relevantly, and that there was a solid case for pursuing the hard line - that the Boolean lattice of classical logic could be modified not just in the orthologic way but more satisfactorily in the relevant algebraic way by removing the paradox $a \leqslant b \cup \overline{b}$ and so $a \leqslant a \cap 1$.

> assertions which are false by the canons of classical logic, meaningless according to the standard version of quantum mechanics, and which nevertheless are both meaningful and true ([45], p. 47)[26]

Let us examine in detail the second, and briefest, of the examples: the others will turn out to succumb to the same treatment. A high-precision determination of the angular momentum J of a diatomic molecule gives the result $J = 0$. The range of the azimuthal angular coordinate of the molecular axis is divided into n equals cells: I_1 $(0 \leqslant \Theta \leqslant \delta\Theta)$, I_2 $(\delta\Theta \leqslant \Theta \leqslant 2\delta\Theta)$, ..., where $\delta\Theta = 2\Pi/n$. Then

(1) Θ is in $I_1 \vee \Theta$ is in $I_2 \vee \ldots \vee \Theta$ is in I_n.

But also, for each molecule, it is false that

(2) $J = 0$ & Θ is in I_j ,

for each j, $1 \leqslant j \leqslant n$.

This is Finkelstein's "real-life" case. But classically (1) and (2) lead to inconsistency and collapse. How is the case handled ultramodally? Let us bring out the classically inconsistent assumptions by abbreviating the argument with q representing $J = 0$ and p_j representing: Θ is in I_j. Then

(1) $p_1 \vee p_2 \vee \ldots \vee p_n$

(2) $\sim(q \,\&\, p_j)$ for each j, and

(3) q.

By (2),

(4) $\sim q \vee \sim p_j$, for each j.

Classically and modally, (3) and (4) entail $\sim p_j$ for each j, by Disjunctive Syllogism, whence

(5) $\sim p_1 \,\&\, \sim p_2 \,\&\, \ldots \,\&\, \sim p_n$,

contradicting (1). But it is evident that ultramodally the argument is invalid, since it applies Disjunctive Syllogism. So there is no problem in admitting the case ultramodally; no special quantum logic is needed. (Observe that it is enough that Disjunctive Syllogism is rejected; it is <u>not</u> required that Material Detachment (γ) be rejected - though easily enough effected - since the premisses (3) and (4) are not provided as logical theses.) Exactly the same points apply to Finkelstein's first and third cases. (For the third, let q represent 'p_x is in I' and p_j represent 'x is J_j', and then the case

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[26] Finkelstein attaches to his article, 'The physics of logic', the apposite aphorism from Wittgenstein: Logic is ultraphysics. If only he had said: Physics is ultralogic.

is as before. The first item, though again relying on Disjunctive Syllogism, calls for more elaborate symbolisation: let p_x^+ represent '$\sigma_x = +h/2$', p_x^- '$\sigma_x = -h/2$' and similarly for p_y^+ and p_y^-. Then the premisses supplied by the Stern-Gerlach experiment Finkelstein presents are: p_x^+, $p_y^+ \vee p_y^-$, $\sim(p_x^+ \,\&\, p_y^+)$, $\sim(p_x^+ \,\&\, p_y^-)$. Inconsistency ensues again classically using Disjunctive Syllogism.) Similar points apply in the case of most of Putnam's examples. (To document cases: Putnam's first example, [46], p. 179, which is simply showing off orthologic, applies the paradox principle $p \,\&\, r \leftrightarrow p$ where r is true; the detailed example on p. 183 is just like Finkelstein's examples and classically would apply Disjunctive Syllogism, and likewise for the sketched case on p. 186.)

Substantiating the second point - that the criticisms that are supposed to tell against Distribution in fact depend on taking over in an unwarranted way negation features of classical logic - involves rather more ado, since in the showdown a straight appeal back to features read off Hilbert space may be attempted. Consider, however, the alleged counterexamples to Distribution. Finkelstein's main counterexample (e.g. in [45], p. 57) in fact uses a class calculus and depends on the paradox, $A \,\&\, D \leftrightarrow D$, where D is true. Moreover the example suppresses all the essential probability details that go into the argument. Putnam's counterexample to Distribution ([46], pp. 180-81) is more fully developed, though again, it seems, insufficiently. The argument looks at the celebrated two-slit experiment. An examination of the argument reveals, however, that it depends not just on Distribution, but on other principles as well. The way in which the conceptual problems of the two-slit experiment arise, not just, or at all, from Distribution, but from other paradoxes of classical logic has been nicely brought out by Mittelstaedt (see the discussion in [53], p. 398). Let A_1 represent the assertion 'The photon in question passes through slit 1', A_2 'The photon in question passes through slit 2', and B 'The photon in question arrives somewhere on the screen'. Mittelstaedt's point, which tells against many-valued approaches, is that the principle of excluded middle, $A_1 \vee \sim A_1$, is certainly valid, but that what the classical argument assumes in the two-slit experiment,

$$(6)\quad B \rightarrow .\ (B \,\&\, A_1) \vee (B \,\&\, A_2)\ ,$$

is quantum-theoretically incorrect. To arrive at (6), moreover, one applies not just Distribution, but a paradox of implication, namely $B \rightarrow .\ B \,\&\, (A_1 \vee \sim A_1)$. Only then does (6) follow using Distribution. Thus an analysis of the two-slit experiment without anomalies does not require rejection of Distribution. None of this appears in Putnam's example, where B is neglected entirely.

Suppose, however, contrary to the evidence, that B could be detached. Putnam's conclusion is still not inevitable. For his argument also relies on repeated applications of the principle that the probability that the photon in question passes through slit 1 and passes through slit 2 (and anything else as well) is zero, i.e., in Putnam's symbols $P(A_1 \,\&\, A_2) = P(A_1 \,\&\, A_2 \,\&\, Q) = 0$. But even if A_1 and A_2 are physically incompatible, as Putnam assumes, that is, even if it is physically impossible that the photon passes through both slit 1 and slit 2, it does not follow, on ultramodal probability

logic, that the logical probability that it does so is zero. I put it to you that, should matters ever come to such a point, this way of breaking the classical argument is at least as plausible as the rejection of Distribution.[27] There remain of course other moves that can be fallen back upon, e.g. the dialectical strategy of simply allowing that some photons pass through both slits, that photons are, at once, both particle-like and wave-like. In fact, the options for description of quantum phenomena remain alarmingly open.

The appeal of an orthological resolution of quantum anomalies, as distinct from a resolution by way of another non-classical logic, goes back to the intimate connections of orthologic with the mathematical formalism of quantum theory in terms of Hilbert space and its subspaces. In order to present a convincing case for the ultralogical way of doing quantum theory, the mathematics of Hilbert spaces will have, eventually, to be relevantly recast. And this involves recasting analysis. Thus an ultramodal quantum theory is going to require for its underpinning an ultramodal analysis - which takes us back to the larger ultramodal program awaiting development.

§14. The way ahead. Very much remains to be done, far more than can be achieved in standard research project lifetimes by the small band of researchers currently working on relevant and ultramodal logics. Perhaps the formulation and initiation of an ultralogical program will spur research in one or more of the following ways: by attracting some to join and foster the ultralogical program; by encouraging the formulation and development of rival relevant ventures; and by inciting opposition to the program from some who develop the program with a view to refuting or undermining it. I invite the reader to participate in this exciting program, if not by joining it, then by beating it.[28]

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27 It is not evident that someone committed, as Putnam is, to quantum logic is entitled to apply classical probability logic. Orthologic should generate its own non-classical probability logic, as critics of Popper have pointed out (see Jammer [53]).

28 This paper was presented, in part, at the Third Latin American Symposium on Mathematical Logic, held at Campinas, Brasil, July 1976.

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RLR R. Routley and R.K. Meyer, Relevant Logics and their Rivals, RSSS, Australian National University (1977).

SC L. Goddard and R. Routley, The Logic of Significance and Context. Volume 1. Scottish Academic Press, Edinburgh (1973).

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Entailment (vol. I) errata

What follows is a list of all errors in vol. I of Anderson and Belnap's Entailment: the logic of relevance and necessity which have been reported as of April 5, 1977. Some of the unimportant ones may be described a trifle vaguely.

Conventions. A "*" in the margin indicates an error which has caused or might cause confusion. Starting at the margin, page number comes first. Then, following a comma, a line number, e.g., "12, 15" referring to page 12, line 15 from top. Or e.g., "95, 7b", referring to page 95, line 7 from bottom. An arrow (-->) means "is to be changed to". A lower case s is sometimes used for the "section" squiggle, as in s8.2. Subscripts are sometimes to be inferred from comparison with the printed text, as in the corrections to 68, 4b.

Thanks to the following for catching many of the errors listed below: Airaksinen, Desrosiers, Meyer, Mourlatos, McRobbie, Przymusinska, Rennie, Rickey, Seldin, Urquhart, Vaughan, Wolf. Thanks to the University of Pittsburgh Computer Center for the use of their facilities in maintaining this list.

Please send additional errors to Nuel Belnap, Department of Philosophy, University of Pittsburgh, Pittsburgh, PA 15260.

Title page. Badly alphabetized

vii, VIII. Strengen --> strenge. (Do not ask how this error crept in. Ever.)

ix, s8.2. strengen --> strenge.

xvi, 36.2. relevance --> relevant

xvii, VIII. strengen --> strenge

xvii, section 41. Boldface: E

xxvii, 9. of systems for --> for

xxxi, 11. losik --> Losik

4, line 6 above 2nd display.

propaeduetic --> propaedeutic

5, 13b (b counts lines from bottom). Jaskowski gets an accent on his first s.

7, 14. P --> P

11, 20. Lightface: F

12, 15. Italic: of

27, running title. Insert box in front of the first A.

29. Ditto.

* 34, PROOF, first line. If a variable a --> Choose a. If it

* 34, 5. above --> above (P. 23)

47, 13. Italic: A and B (twice each).

* 47, 10b. we saw in s5 --> one can see

* 61, last formula of display. Insert an arrow.

68, 5. Elimination theorem --> elimination rule ER

68, 4b. An --> Ak . Cn --> Cm

68, 2b. An --> Ak

68, 1b. Cn --> Cm

69, 4. Cn --> Cm

69, second display. Close up dots.

69, 13b. cognation classes --> cognate sequents

72, s8.2. strensen --> strense

83, (1). Boldface: E

84, second theorem. Put some marks on the names. Also in next paragraph.

95, 7b. Boldface: Greek letter Pi

104, THEOREM 4, line 1. Italic: A

* 108, 2b. Delete the little curl.

124, 12. Delete the funny mark in front of the P.

125, 4b. Backwards c --> o

149, 3b. Boldface italic: 1. Lightface italic: A

161, 11-12. Subscript the i and i' (three times).

170, 13b. Delete funny mark over s.

* 175, 4. First i --> j

182, 4. Delete line through z.

188, 14b. equivalences --> equivalence

194, 7. Bialnicki --> Bialynicki

206, 1b. S --> s

213, 2b. Improve the dot on the i.

232, 5b. Boldface: E

235, 2. Italic: A, B

238, bottom display. Insert vertical line.

259, 6b. Italic: B. Also A in 3b. In last line, hoort --> oort.

* 261, 2. 23.5 --> 23.6

* 265, 9. An --> Am

* 265, 10. n>0 --> m≥1

265, 12. An --> Am

265, 13. An --> Am

* 269, line above D1. Delete box on E.

271, 9. Subscript the +.

271, 10. Boldface: R

271, s23, line 6. Straighten: df

* 277, s23.6, line 1. 22 --> 22.2.1

290, last display, d. Star in wrong font.

303, bottom display, E7'. Insert initial parenthesis.

* 307, table for V*, extreme bottom right corner. Delete the bar.

310, 5. Insert right parenthesis to left of arrow.

* 311, Theorem 5, (iv). Turnstile needs two horizontal lines, as in (ii) near bottom of page 304.

312, A1'. Add right parenthesis at end.

312, A2'. if --> If

316, 15. Hyphens are too short.

316, 14b. Delete initial parenthesis.

* 331, first display. The backward turnstile should be a forward turnstile.

331, 12b. Italic: from

332, 2. Italic: of, from

* 334, 3rd line above matrix. Insert left parenthesis after box.

340, A17. Delete final dot.

342, below display. Delete wandering marks.

343, 2. Boldface: E

344, 5. Boldface: R

* 344, 11. 23.2.2 --> 22.3.3

345, 13b. Italic: A (twice).

347. This discussion needs changing to refer more uniformly to FRM and FEM.

353, 5b. Roman: a

* 355, 2. (:a --> (a

* 371, 6b. 14 --> 13

375, 10. Umlaut the U.

383, below display in long paragraph. Put parentheses around names of seven rules.

* 384, rule (o turnstile). ((--> I((

* 393, 4. E --> E with subscript arrow. Also, formulas --> pure arrow formulas

393, 10b. etre --> etre (with hat on first e)

400, iii, b. Make first minus lighter.

404, 7b. Boldface italic: R

414, 3b. Delete: just defined,

428, 18b. Insert right parenthesis before period.

430, 3. Perry's --> Parry's

433, display 12. or --> (or

466, 15-17. Boldface: E (three times).

466, 2b. Ditto.

467, 1. Ditto.

469. More space between lines (w) and (x).

* 470, 14b. application --> implication

471, paragraph beginning "A last word," line 4. 48.5.3 --> 51.3. Next line: of R --> and R

473, 1. P --> p

487, 10b. Extra space around a and b.

491, 19b. P --> p

493, 7. .) -->),

496. BIALNICKI --> BIALYNICKI

499, Feys and Ladriere. Turn the accent around.

504, LUKASIEWICZ 1929. Elementary --> Elementy

506, 5. 55 --> 53

513, 4. 357 --> 37

513, Whately. Supply publisher.

515, 5. Symbolic --> Symbolic logic

515, Meyer 1974. Implicatiom --> Implication

515. Insert references for WAJSBERG 1937, GEACH 1958, DANTZIG 1930

517, 7. the --> then

517. +3 --> xxxi (three times). (This error, and others below, were due to a communication lapse. They were not caught because the authors unwisely decided not to

proofread the index.)

518, 1. Bialnicki --> Bialynicki

519, Jaskowski. Accent on first s.

520. Alphabetize each Polish l after l. Also, reference from Marcus to Barcan Marcus. In second column, +3 --> xxxi. And Mihailescu gets his a checked.

520, 1. Accent on s in Los (as on page 190, line 11).

521, Rasiowa. "and Sikorski" starts subentry. Schutte gets umlaut.

522, +3 --> xxxi. van Heijenhoort --> van Heijenoort

523, 7. auotmatic --> automatic

523, assignment. de --> De

524, CFL. Boldface system names.

525, co-tenability. Insert as subentry: necessity and, 269

525, deduction theorem. Delete: for Efdf, 211

* 527, E-->1, 142. Put curl on top of arrow. Ditto on next line.

528, 13. SEE --> See

528, fallacies of modality, S5. Boldface: 5

529, fourteen. Insert: xxix.

529, FS4-->. Boldface: 4

530. Insert: idempotence, 421

530, intensional lattices, implicational operator on. Insert: 285-286

530, Jaskowski theorem. Accent on first s.

531, Lesniewski-Mihailescu theorem. Insert: 87-88

532, matrices for:. Insert as subentry: tautological entailments, 161-162

532, natural deduction. Insert as subentry: survey, 346-348

533, notation for. Put parentheses around: See...Index

533, second column. *4 --> xxiv. *5 --> xxv. *5-7 --> xxv-xxvii.

534. Insert: Positive rigorous logic, 294

537, RM0-->. Boldface: 0

537, RM3. Boldface: 3

537, second column. Boldface system names.

537, Salmon's. principal --> principle

538. strengen --> strenge

538, strict implication. Boldface system names.

538, subformula theorem. Boldface: C

538, SZ. Insert: 400. Also, in SZ*: 421 --> 400

541, grammatical symbols, double arrow. 9 --> 91

541, proof-theoretical symbols. The T should be a subscript.

* 541, algebraic symbols. V --> inverted V. Also: bound --> bound; meet

* 542, 1. Inverted V --> V. Also: bound --> bound; Join

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